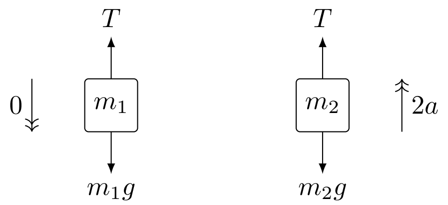
4801. Assume, for a contradiction, that $\log_a b$ is rational, and can be expressed, for $p, q \in \mathbb{Z}$, as $\log_a b = p/q$. Exponentiating both sides over base a, $b = a^{p/q}$. Raising both sides to the power q, we reach

$$b^q = a^p.$$

Both sides are integers. The LHS has no factors of a, so the RHS must also have no factors of a. Hence, $p = 0$, giving $\log_a b = 0$. But $\log_a b \neq 0$, since $b \neq 1$. This is a contradiction. Therefore, if a and b are coprime, then $\log_a b$ is irrational. □

4802. Call the upwards acceleration of the pulley a. If the bob of mass m_1 remains at rest, then the bob of mass m_2 must be accelerating upwards at $2a$. The force diagrams are



So, $T = m_1 g$ and $m_2 g - T = 2m_2 a$. Solving these,

$$a = \frac{m_2 g - m_1 g}{2m_2}.$$

If instead we keep the bob of mass m_2 at rest, then the acceleration of the pulley must be downwards rather than upwards. Other than this, the problem is the same. So, the (signed) upwards accelerations that keep each bob at rest are

$$\text{Bob of mass } m_1 \text{ at rest}: \quad a_1 = \frac{m_2 g - m_1 g}{2m_2},$$
$$\text{Bob of mass } m_2 \text{ at rest}: \quad a_2 = \frac{m_1 g - m_2 g}{2m_1}.$$

4803. Let the functions have definitions $\text{f}(x) = ax + b$ and $\text{g}(x) = cx + d$. Applying f three times,

$$\begin{aligned} & \text{f}(x) = ax + b \\ \implies & \text{f}^2(x) = a(ax + b) + b \\ & \qquad \equiv ax^2 + ab + b \\ \therefore\ & \text{f}^3(x) = a(ax^2 + ab + b) + b \\ & \qquad = a^3x + a^2b + ab + b. \end{aligned}$$

Stating the same result for $\text{g}^3(x)$, we know that

$$a^3x + a^2b + ab + b \equiv c^3x + c^2d + cd + d.$$

Equating coefficients of x gives $a^3 = c^3$. Since there is only one cube root, this implies $a = c$. Substituting this back into the constant terms,

$$\begin{aligned} & a^2b + ab + b = a^2d + ad + d \\ \implies & b(a^2 + a + 1) = d(a^2 + a + 1). \end{aligned}$$

The quadratic factor has $\Delta = -3 < 0$, so can't be zero. Hence, we can divide by it, giving $b = d$. So, f and g are the same function, as required.

4804. Written longhand, the graph is

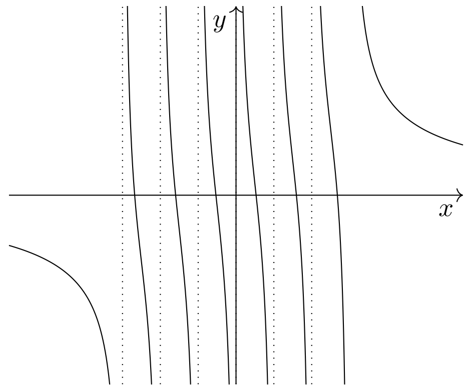
$$y = \frac{1}{x+3} + \frac{1}{x+2} + ... + \frac{1}{x-3}.$$

This has asymptotes at $x = -3, -2, ..., 3$. At each, there is sign change. Consider the behaviour either side of the asymptote $x = p$.

① As $x \to p^-$, the dominating asymptotic term is negative, so $y \to -\infty$.

② As $x \to p^+$, the dominating asymptotic term is positive, so $y \to \infty$.

There is also a horizontal asymptote at $y = 0$. So, the curve is



4805. Let the outer pentagon have unit side length. A regular pentagon has interior angles of 108° and exterior angles of 72°.

① The isosceles triangles with one vertex on the outer pentagon have angles (72°, 72°, 36°).

② The isosceles triangles with two vertices on the outer pentagon have (36°, 36°, 108°).

Bisecting one of the latter isosceles triangles gives a right-angled triangle with angle 36° and adjacent side $\frac{1}{2}$. Its hypotenuse is $\frac{1}{2}\sec 36°$. Bisecting one of the former, we have a right-angled triangle with angle 18° and hypotenuse $\frac{1}{2}\sec 36°$. Its opposite is $\frac{1}{2}\sec 36° \sin 18°$. The side length of the smaller pentagon is twice the above, which is

$$\sec 36° \sin 18°.$$

Consider the double-angle result

$$\sin 72° = 2\sin 36° \cos 36°.$$

We can rearrange this to

$$\begin{aligned} \sec 36° &= \frac{2\sin 36°}{\sin 72°} \\ &= \frac{2\sin 36°}{\cos 18°} \\ &= \frac{4\sin 18° \cos 18°}{\cos 18°} \\ &= 4\sin 18°. \end{aligned}$$

So, the ratio of side lengths is $1 : 4\sin^2 18°$.

4806. Let $u = \ln x$ so that $x = e^u$ and $dx = e^u\, du$. The new limits are $-\infty$ to 0. Enacting the substitution,

$$\begin{aligned}&\int_0^1 (\ln x)^3\, dx \\ =\ &\int_{-\infty}^0 u^3 e^u\, du.\end{aligned}$$

We use the tabular integration method:

Signs	Derivatives	Integrals
$+$	u^3	e^u
$-$	$3u^2$	e^u
$+$	$6u$	e^u
$-$	6	e^u
$+$	0	e^u

This gives

$$\int u^3 e^u\, dx = (u^3 - 3u^2 + 6u - 6)e^u + c.$$

So, the indefinite integral is

$$\Big[(u^3 - 3u^2 + 6u - 6)e^u\Big]_{-\infty}^0.$$

As $u \to -\infty$, the exponential e^u dominates the polynomial factor, so that the lower limit is zero. The upper limit is -6. So,

$$\int_0^1 (\ln x)^3\, dx = -6, \text{ as required.}$$

4807. Let the x intercepts be $\{b-d, b, b+d\}$. Apply the following transformations to the curve:

① translate by $-b\mathbf{i}$, so that the x intercepts are $\{-d, 0, d\}$,

② stretch by scale factor $\frac{1}{d}$ in the x direction, so that the x intercepts are $\{-1, 0, 1\}$,

③ stretch by some ($\pm$) scale factor in the y direction, so that the x intercepts remain $\{-1, 0, 1\}$ and the graph is now monic.

By the factor theorem, a monic cubic $y = \mathrm{f}(x)$ with x intercepts at $\{-1, 0, 1\}$ is

$$\begin{aligned}y &= (x+1)x(x-1) \\ &\equiv x^3 - x.\end{aligned}$$

The result is proved by construction.

4808. (a) The values are

x	0.1	0.2	0.3
$\cos x$	0.995004	0.980067	0.955336
$\mathrm{c}_2\, x$	0.995	0.98	0.955

(b) We want $\mathrm{c}_4(x) \approx \cos(x)$, so

$$\begin{aligned}\cos x &\approx 1 - \tfrac{1}{2}x^2 + kx^4 \\ &= \mathrm{c}_2(x) + kx^4.\end{aligned}$$

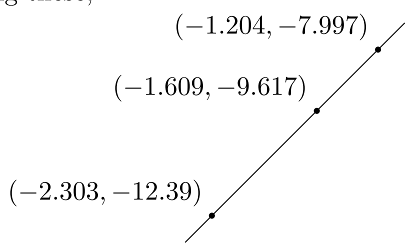
So, $kx^4 \approx \cos x - \mathrm{c}_2(x)$. Taking logs,

$$\ln k + 4\ln x \approx \ln\big(\cos(x) - \mathrm{c}_2(x)\big).$$

This is an approximately linear relationship between $\ln\big(\cos(x) - \mathrm{c}_2(x)\big)$ and $\ln x$.

$\ln x$	-2.303	-1.609	-1.204
$\ln\big(\cos(x) - \mathrm{c}_2(x)\big)$	-12.39	-9.617	-7.997

Plotting these,



Since these three points are approximately collinear, a polynomial relationship holds well. The gradient of the above line, using the outer two points, is $3.997 \approx 4$. So, the polynomial relationship is indeed quartic. Using the point $(-2.303, -12.39)$,

$$\begin{aligned}&\ln k + 4\cdot -2.303 \approx -12.39 \\ \Longrightarrow\ &\ln k \approx -3.178 \\ \Longrightarrow\ &k \approx 0.04166...\end{aligned}$$

The proposed value is $\frac{1}{24} = 0.041\dot{6}$. So, the correct quartic approximation uses $k = \frac{1}{24}$:

$$\mathrm{c}_4(x) = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4.$$

4809. The chain rule gives

$$\begin{aligned}a &= \frac{dv}{dt} \\ &\equiv \frac{dv}{dx} \times \frac{dx}{dt} \\ &= \frac{dv}{dx}v.\end{aligned}$$

So, the DE may be rewritten as

$$v\frac{dv}{dx} = \tfrac{3}{2}x^2.$$

Separating the variables,

$$\begin{aligned}&\int v\, dv = \int \tfrac{3}{2}x^2\, dx \\ \Longrightarrow\ &\tfrac{1}{2}v^2 = \tfrac{1}{2}x^3 + c \\ \Longrightarrow\ &v^2 = x^3 + d \\ \therefore\ \ &|v| = \sqrt{x^3 + d}.\end{aligned}$$

For large x, the constant d becomes negligible. Hence, $|v|$ is approximately proportional to $\sqrt{x^3}$, which is $x^{\frac{3}{2}}$, as required.

4810. Reflection in the line $y = x + c$ is equivalent to reflection in the line $y = x$, followed by translation by vector $-c\mathbf{i} + c\mathbf{j}$.

- Reflection in $y = x$ gives $x = ay^2 + by^2 + c$.
- Translation by $-c\mathbf{i}+c\mathbf{j}$ means replacement of x by $x+c$ and replacement of y by $y-c$. This gives $x + c = a(y - c)^2 + b(y - c) + c$.

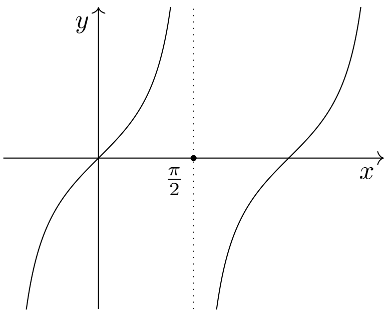
Simplifying, the transformed parabola is

$$x = a(y - c)^2 + b(y - c).$$

4811. (a) The derivative is $\sec^2 x$. So, the gradient of the normal at $x = p$ is $-\cos^2 p$. The equation of the normal is therefore

$$\begin{aligned} y - \tan p &= -\cos^2 p(x - p) \\ \Longrightarrow y &= (p - x)\cos^2 p + \tan p. \end{aligned}$$

(b) The first two branches of $y = \tan x$, over the two domains $(-\pi/2, \pi/2)$ and $(\pi/2, 3\pi/2)$, have a centre of rotational symmetry at $(\pi/2, 0)$:



By symmetry and the shape of the curve, the shortest path between these two branches must pass through $(\pi/2, 0)$, thereby having the form $y = m(x - \pi/2)$.

(c) Substituting the point $(\pi/2, 0)$ into the normal in part (a),

$$0 = \left(p - \tfrac{\pi}{2}\right)\cos^2 p + \tan p.$$

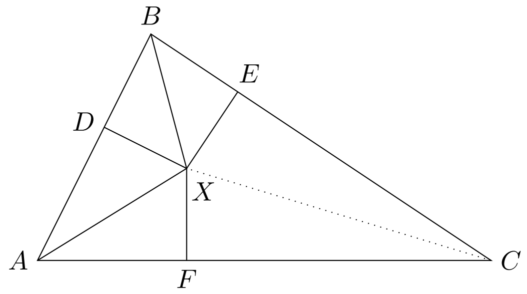
This is not analytically solvable. The Newton-Raphson iteration is

$$\begin{aligned} p_{n+1} &= p_n \\ &- \frac{\left(p_n - \frac{\pi}{2}\right)\cos^2 p_n + \tan p_n}{\cos^2 p_n - \left(p_n - \frac{\pi}{2}\right)\sin 2p_n + \sec^2 p_n}. \end{aligned}$$

With $p_0 = 1$, we get $p_1 = 0.6717$, and then $p_n \to 0.59251$. The shortest distance is

$$\begin{aligned} d &= 2\sqrt{\left(0.59251 - \tfrac{\pi}{2}\right)^2 + \tan^2 0.59251} \\ &= 2.375069... \\ &= 2.375 \text{ (4sf), as required.} \end{aligned}$$

4812. We begin with the angle bisectors at A and B, intersecting at X, together with perpendiculars drawn to all three sides. We need to show that the dotted line CX bisects the angle at C.



Triangles AFX and ADX are congruent, as are BDX and BEX. So, perpendiculars DX, EX, FX are all the same length. Triangles ECX and FCX are right-angled, with the hypotenuse CX in common and $|EX| = |FX|$. Hence, they are congruent, proving that CX bisects angle C. The angle bisectors are therefore concurrent. □

— Nota Bene —

The fact that the three perpendicular bisectors DX, EX, FX are all the same length is what gives point X its name: it is the *incentre* of the triangle, which is the centre of the largest circle which can be inscribed.

4813. Completing the square, the denominator is

$$(x + 4)^2 + 1.$$

So, let $x + 4 = \tan\theta$. This gives $dx = \sec^2\theta\, d\theta$. Enacting the substitution,

$$\begin{aligned} &\int \frac{1}{(x+4)^2 + 1}\, dx \\ =& \int \frac{1}{\tan^2\theta + 1}\sec^2\theta\, d\theta \\ =& \int 1\, d\theta \\ =&\ \theta + c \\ =&\ \arctan(x + 4) + c. \end{aligned}$$

4814. If $y = \mathrm{g}'(x)$ has rotational symmetry around (a, b), then $\mathrm{g}'(a + x)$ and $\mathrm{g}'(a - x)$ are equidistant from b. Put into algebra, this is

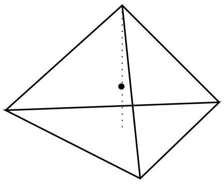
$$\begin{aligned} &\mathrm{g}'(a + x) - b = b - \mathrm{g}'(a - x) \\ \Longrightarrow\ &\mathrm{g}'(a + x) + \mathrm{g}'(a - x) = 2b. \end{aligned}$$

Integrating this, we get a minus sign in the second term by the reverse chain rule:

$$\mathrm{g}(a + x) - \mathrm{g}(a - x) = 2bx + c.$$

So, as required, $y = \mathrm{g}(a + x) - \mathrm{g}(a - x)$ is linear, with gradient $2b$.

4815. The shortest distance between the centre of the tetrahedron and one of its faces is to the centre of that face. Placing the relevant face flat to the ground, this is simply the vertical height of the centre of the tetrahedron.



The total height of the tetrahedron is $\sqrt{6}/3$, as can be found by elementary trigonometry.

The centre of a 2D triangle lies $2/3$ of the way along its medians. By higher-dimensional analogy, the centre of a 3D tetrahedron lies $3/4$ of the way along its medians. So, the height of the centre and thus the radius of the insphere is

$$r = \frac{\sqrt{6}}{3} \times \frac{1}{4} = \frac{1}{\sqrt{24}}, \text{ as required.}$$

4816. (a) Expanding the factors on the RHS,

$$\sin\left(\tfrac{P+Q}{2}\right) \equiv \sin\tfrac{P}{2}\cos\tfrac{Q}{2} + \cos\tfrac{P}{2}\sin\tfrac{Q}{2},$$
$$\sin\left(\tfrac{P-Q}{2}\right) \equiv \sin\tfrac{P}{2}\cos\tfrac{Q}{2} - \cos\tfrac{P}{2}\sin\tfrac{Q}{2}.$$

The product of these is a difference of two squares. So, the RHS of the identity is

$$\begin{aligned} &2\left(\sin^2\tfrac{P}{2}\cos^2\tfrac{Q}{2} - \cos^2\tfrac{P}{2}\sin^2\tfrac{Q}{2}\right) \\ \equiv\ &2\left(\sin^2\tfrac{P}{2}\cos^2\tfrac{Q}{2} - \cos^2\tfrac{P}{2}\left(1-\cos^2\tfrac{Q}{2}\right)\right) \\ \equiv\ &2\left(\cos^2\tfrac{Q}{2} - \cos^2\tfrac{P}{2}\right). \end{aligned}$$

Using double-angle formulae, this is

$$\begin{aligned} &2\left(\tfrac{1}{2}(\cos Q + 1) - \tfrac{1}{2}(\cos P + 1)\right) \\ \equiv\ &\cos Q - \cos P, \text{ as required.} \end{aligned}$$

(b) Rearranging,

$$\cos 6x - \cos 2x + \sin 4x = 0$$

Using the sum-to-product formula,

$$\begin{aligned} &2\sin 4x \sin(-2x) + \sin 4x = 0 \\ \implies &\sin 4x(-2\sin 2x + 1) = 0 \\ \implies &\sin 4x = 0 \text{ or } \sin 2x = \tfrac{1}{2}. \end{aligned}$$

Over the domain $[0, \pi/2]$, the solution is

$$x = 0, \tfrac{\pi}{12}, \tfrac{\pi}{4}, \tfrac{5\pi}{12}, \tfrac{\pi}{2}.$$

4817. We can rule ① and ② out, due to the presence of x and y in *unsquared* form. This means ① and ② do not have the required symmetry: $xy = 1$ is a boundary curve, but $xy = -1$ isn't.

Then ③ and ④ are negatives. Testing the origin, $(0,0)$ satisfies ④ but not ③. Hence, ③ must be the defining inequality.

4818. Consider C_3 and C_4, classifying the result by the number of heads attained in C_3:

C_3	C_4	Probability
0	$1,2,3,4$	$\frac{1}{8} \times \frac{15}{16} = \frac{15}{128}$
1	$2,3,4$	$\frac{3}{8} \times \frac{11}{16} = \frac{33}{128}$
2	$3,4$	$\frac{3}{8} \times \frac{5}{16} = \frac{15}{128}$
3	4	$\frac{1}{8} \times \frac{1}{16} = \frac{1}{128}$

The total probability is $1/2$. This result generalises. To prove it, consider C_{2k-1} and C_{2k}. The key fact is the symmetry of the table above and below the central line, which occurs between k and $k+1$.

C_{2k-1}	C_{2k}	Probability
0	$1, ..., 2k$	$a_0 \times b_0$
1	$2, ..., 2k$	$a_1 \times b_1$
...	...	...
$k-1$	$k, ..., 2k$	$a_k \times b_k$
k	$k+1, ..., 2k$	$a_k \times (1-b_k)$
...	...	...
$2k-2$	$2k-1, 2k$	$a_1 \times (1-b_1)$
$2k-1$	$2k$	$a_0 \times (1-b_0)$

Pairing the entries,

$$\begin{aligned} &a_0 \times b_0 + a_0 \times (1-b_0) \\ +\ &a_1 \times b_1 + a_1 \times (1-b_1) \\ +\ &... \\ +\ &a_k \times b_k + a_k \times (1-b_k) \\ \equiv\ &a_0 + a_1 + ... + a_k. \end{aligned}$$

This is the sum of half (a symmetrical half) of the probabilities of the distribution $B(2k, 1/2)$. So, the probability is $1/2$.

4819. The equations of the normals are

$$\begin{aligned} y &= -\tfrac{1}{2p}(x-p) + p^2, \\ y &= -\tfrac{1}{2(p+2)}(x-p-2) + (p+2)^2. \end{aligned}$$

These have the same y value at $x = \frac{15}{2}$:

$$\begin{aligned} &-\tfrac{1}{2p}\left(\tfrac{15}{2} - p\right) + p^2 \\ &\qquad = -\tfrac{1}{2(p+2)}\left(\tfrac{15}{2} - p - 2\right) + (p+2)^2. \end{aligned}$$

Simplifying a little,

$$\tfrac{1}{p}\left(15 - 2p\right) = \tfrac{1}{p+2}\left(15 - 2p - 4\right) - 16p - 16.$$

Multiplying by $p(p+2)$,

$$\begin{aligned} (15-2p)(p+2) = p(15-2p-4) \\ - 16p(p+1)(p+2). \end{aligned}$$

This gives

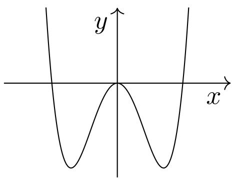
$$\begin{aligned} &16p^3 + 48p^2 + 32p + 30 = 0 \\ \implies &p = -\tfrac{5}{2}. \end{aligned}$$

4820. (a) Let $y = \mathrm{h}(x)$ be the original tangent at $x = a$. The original equation for re-intersections is $\mathrm{f}(x) = \mathrm{h}(x)$. Because g (and the operation of differentiation) is linear, the tangent line to $y = \mathrm{f}(x) + \mathrm{g}(x)$ at $x = a$ is $y = \mathrm{h}(x) + \mathrm{g}(x)$. The new equation for re-intersections will be

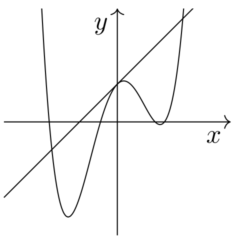
$$\begin{aligned} & \mathrm{f}(x) + \mathrm{g}(x) = \mathrm{h}(x) + \mathrm{g}(x) \\ \Longleftrightarrow\ & \mathrm{f}(x) = \mathrm{h}(x). \end{aligned}$$

So, the x coordinates of the re-intersections will be the same either way.

(b) Let $\mathrm{g}(x) = -x - 1$. Transforming as in part (a), the quartic is $y = x^4 - 3x^2$. This has even symmetry. Hence, the tangent at $x = 0$ will re-intersect the curve symmetrically at $x = \pm b$, but no other tangent will.



The tangent line is the x axis. Reversing the transformation, the original tangent line was $y = x + 1$.



4821. Using the cosine rule,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

This gives

$$\begin{aligned} \sin C &= \sqrt{1 - \cos^2 C} \\ &= \sqrt{1 - \frac{(a^2+b^2-c^2)^2}{4a^2b^2}} \\ &\equiv \frac{\sqrt{4a^2b^2 - (a^2+b^2-c^2)^2}}{2ab}. \end{aligned}$$

So, the area of the triangle is

$$\begin{aligned} A_\triangle &= \tfrac{1}{2}ab\sin C \\ &= \tfrac{1}{4}\sqrt{4a^2b^2 - (a^2+b^2-c^2)^2} \\ &\equiv \tfrac{1}{4}\sqrt{-a^4 - b^4 - c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2}. \end{aligned}$$

Factorising this, we get

$$\tfrac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

Bringing the $1/4$ into the square root, we distribute a factor of $1/2$ to each bracket. This gives

$$A_\triangle = \sqrt{s(s-a)(s-b)(s-c)}, \text{ as required.}$$

4822. The quartic is stationary at $x = p, q, r$. So, its derivative has roots $x = p, q, r$, and must therefore be a scalar multiple of $ax^3 + bx^2 + cx + d$:

$$\frac{dy}{dx} = kax^3 + kbx^2 + kcx + kd.$$

Integrating this,

$$y = \tfrac{1}{4}kax^4 + \tfrac{1}{3}kbx^3 + \tfrac{1}{2}kcx^2 + \tfrac{1}{2}kdx + e.$$

The quartic is monic, so $\frac{1}{4}ka = 1$, giving $k = \frac{4}{a}$. Also, since the curve passes through the origin, $e = 0$. This gives the quartic as

$$y = x^4 + \tfrac{4b}{3a}x^3 + \tfrac{2c}{a}x^2 + \tfrac{4d}{a}kx.$$

4823. We integrate by parts. Let $u = x$ and $v' = e^x \cos x$. This gives $u' = 1$. To find v, we need to integrate by parts. We use the tabular integration method:

Signs	Derivatives	Integrals
$+$	e^x	$\cos x$
$-$	e^x	$\sin x$
$+$	e^x	$-\cos x$

This gives

$$\begin{aligned} & v = e^x \sin x + e^x \cos x - v \\ \Longrightarrow\ & v = \tfrac{1}{2}e^x(\sin x + \cos x). \end{aligned}$$

So, the original integral is

$$\begin{aligned} & \int xe^x \cos x\,dx \\ &= \tfrac{1}{2}xe^x(\sin x + \cos x) - \tfrac{1}{2}\int e^x(\sin x + \cos x)\,dx. \end{aligned}$$

We already have the integral of $e^x \cos x$. Let

$$I = \int e^x \sin x\,dx.$$

Signs	Derivatives	Integrals
$+$	e^x	$\sin x$
$-$	e^x	$-\cos x$
$+$	e^x	$-\sin x$

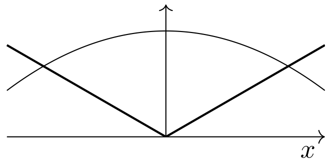
This gives

$$\begin{aligned} & I = -e^x \cos x + e^x \sin x - I \\ \Longrightarrow\ & I = \tfrac{1}{2}e^x(\sin x - \cos x). \end{aligned}$$

Putting all of this together,

$$\begin{aligned} & \int xe^x \cos x\,dx \\ &= \tfrac{1}{2}xe^x(\sin x + \cos x) - \tfrac{1}{2}\int e^x(\sin x + \cos x)\,dx \\ &= \tfrac{1}{2}xe^x(\sin x + \cos x) - \tfrac{1}{4}e^x(\sin x + \cos x) \\ &\qquad - \tfrac{1}{4}e^x(\sin x - \cos x) + c \\ &\equiv \tfrac{1}{2}e^x\big(x\sin x + x\cos x - \sin x\big) + c. \end{aligned}$$

4824. The relevant graphs are



The projectile bounces, for $x > 0$, where

$$\begin{aligned}\tfrac{\sqrt{3}}{3}x &= \frac{u^2}{2g} - \frac{gx^2}{2u^2} \\ \Longrightarrow \frac{gx^2}{2u^2} + \tfrac{\sqrt{3}}{3}x - \frac{u^2}{2g} &= 0.\end{aligned}$$

Taking the positive root in the quadratic formula,

$$\begin{aligned}x &= \frac{-\frac{\sqrt{3}}{3} + \sqrt{\frac{1}{3}+1}}{\frac{g}{u^2}} \\ &\equiv \frac{u^2}{\sqrt{3}g}.\end{aligned}$$

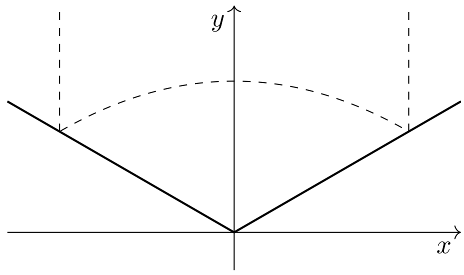
The gradient of the trajectory is

$$\frac{dy}{dx} = -\frac{g}{u^2}x.$$

Evaluating at the point at which the projectile bounces, the gradient of the trajectory is

$$\begin{aligned}m &= -\frac{g}{u^2} \times \frac{u^2}{\sqrt{3}g} \\ &= -\tfrac{\sqrt{3}}{3}.\end{aligned}$$

So, the angle of inclination when it bounces is 30° below horizontal. The surfaces are inclined at 30°. And there is no loss of speed, so, when it bounces, the path of the projectile is reflected in the normal to the surface. So, the projectile travels vertically upwards after bouncing. Symmetry dictates that the next bounce puts the projectile back on its original trajectory, and so on.



The motion is periodic, as required.

4825. Firstly, we simplify a little:

$$1 - \frac{1}{r^2} \equiv \frac{r^2-1}{r^2} \equiv \frac{(r+1)(r-1)}{r^2}.$$

By the chain rule,

$$\begin{aligned}\frac{dy_n}{dy_1} &\equiv \frac{dy_n}{dy_{n-1}} \times \frac{dy_{n-1}}{dy_{n-2}} \times \ldots \times \frac{dy_2}{dy_1} \\ &= \frac{(n+1)(n-1)}{n^2} \times \frac{n(n-2)}{(n-1)^2} \times \ldots \times \frac{3\cdot 1}{2^2}.\end{aligned}$$

Most factors cancel, with e.g. $(n-1)^2$ appearing in the numerator of the first and third factors, and the denominator of the second. The only factors uncancelled are at the beginning and the end:

$$\begin{aligned}\frac{dy_n}{dy_1} &= \frac{n+1}{n} \times \frac{1}{2} \\ &\equiv \frac{n+1}{2n}.\end{aligned}$$

Integrating this with respect to y_1,

$$y_n = \frac{n+1}{2n}y_1 + c, \text{ as required.}$$

4826. Since ○ went first, there are five ○ and four × in the grid. Clearly, for a draw, the middle right must be ×. Labelling the remaining boxes, we have

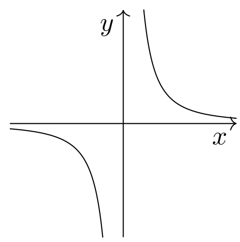
×	1	2
○	○	×
3	4	5

We classify by number of ○ in the top row.

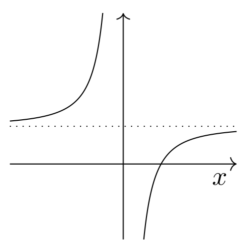
- NONE. In this case, 3, 4, 5 must all contain ○. This makes three in a row. So, this option contributes no grids.
- ONE. There are two options:
  - ○ in 1. There must be × in both 2 and 4. This leaves ○ in 3 and 5. This contributes one grid.
  - ○ in 2. There must be × in both 1 and 3. This leaves ○ in 4 and 5. This contributes one grid.
- TWO. There must be × in both 3 and 4. This leaves ○ in 5. This contributes one grid.

So, there are three grids in total, as required.

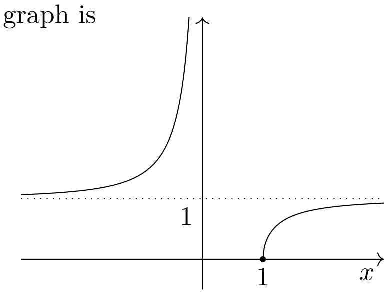
4827. Consider $y = x^{-1.8} = x^{-\frac{9}{5}}$. Since, both 9 and 5 are odd, this is a curve akin to $y = x^{-1}$ and $y = x^{-3}$, with gradients in between the two:



The curve $y = 1 - x^{-1.8}$ is then



The x intercept in the above graph is $(1, 0)$; y is negative for $x \in (0, 1)$. So, the original graph has no points over this domain. The tangent at $x = 1$ becomes parallel to the y axis, in the manner of $y = \sqrt{x}$ at the origin. Taking the square root, the y values remain above and below the dotted line. So, the graph is



4828. The probability that couple A sits together is $2/5$: place A_1 wlog and then see if A_2 sits next door. So, in a large number of trials, we would expect to see, on average, couple A sitting together in 2 out of every 5 trials. The same is true of couples B and C. The expectation of each couple's presence is $2/5$, so the expected number of couples present is $3 \times 2/5 = 6/5$.

This can seem a little like magic. So, below is the brute force version...

——— Alternative Method ———

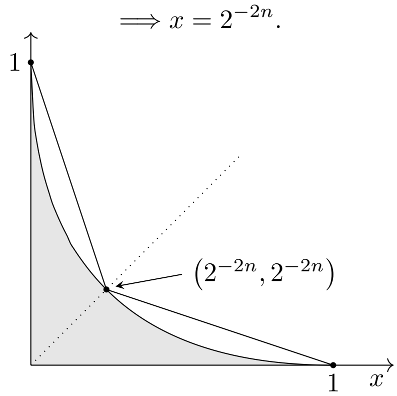
The possibilities for the number of couples sitting together are $\{0, 1, 2, 3\}$:

- *3 couples sitting together.* Place A_1 wlog. The probability that A_2 sits next door is $2/5$. Then put someone next to A_2, call them B_1. The probability that B_2 sits next to B_1 is $1/3$. If the As and Bs are successful, the Cs are guaranteed to be. So, the probability that three couples sit together is
$$\tfrac{2}{5} \times \tfrac{1}{3} = \tfrac{2}{15}.$$
- *2 couples sitting together.* There are three choices for the couple not sitting together. Place A_1 wlog. The probability that A_2 sits opposite is $1/5$. This leaves two spaces of two seats each. Place B_1 wlog. The probability that B_2 sits next door is $1/3$. If they do, the Cs are then sorted. So, the probability that two couples sit together is
$$3 \times \tfrac{1}{5} \times \tfrac{1}{3} = \tfrac{3}{15}.$$
- *1 couple sitting together.* There are three choices for that couple. Choose the As. Place A_1 wlog. The probability that A_2 sits next door is $2/5$. Place someone next to them, and call them B_1. There is only one successful seat for B_2, with probability $1/3$. So, the probability that exactly one couple sits together is
$$3 \times \tfrac{2}{5} \times \tfrac{1}{3} = \tfrac{6}{15}.$$

Hence, the expectation of the number of couples sitting together is
$$\begin{aligned}&3 \times \tfrac{2}{15} + 2 \times \tfrac{3}{15} + 1 \times \tfrac{6}{15}\\ &= \tfrac{6}{5}, \text{ as required.}\end{aligned}$$

4829. The boundary equations, for $n = 1, 2, 3$ are
$$\begin{aligned}x^{\frac{1}{2}} + y^{\frac{1}{2}} &= 1,\\ x^{\frac{1}{4}} + y^{\frac{1}{4}} &= 1,\\ x^{\frac{1}{6}} + y^{\frac{1}{6}} &= 1.\end{aligned}$$

These equations are only defined in the positive quadrant. So, the regions in question are bounded by the axes. The region A_1 is as shown below. The other regions are broadly the same, with positive curvature everywhere on $(0, 1)$. The area of each region can be bounded by finding the intersection with $y = x$. In the case of $n = 1$, this is at $(1/4, 1/4)$. In general, the intersection with $y = x$ lies at
$$\begin{aligned}&2x^{\frac{1}{2n}} = 1\\ \Longrightarrow\ &x = 2^{-2n}.\end{aligned}$$



So, the area A_n is, in each case, bounded by the area B_n below and to the left of the line segments shown. The area B_n is 2^{-2n}. So, as $n \to \infty$, $B_n \to 0$. And, since A_n is squeezed between 0 and B_n,
$$\lim_{n \to \infty} A_n = 0.$$

——— Nota Bene ———

The technical name for the result invoked above is the *squeeze theorem.*

4830. The leading coefficient of P_1 is -1. So, before its enlargement, P_2 is a monic parabola, of the form $y = x^2 + a$. To enlarge by factor $\frac{1}{2}$, we replace x by $2x$ and y by $2y$. This gives a parabola of the form $2y = (2x)^2 + a$, which we can rewrite as
$$y = 2x^2 + b.$$

For intersections with $y = -x^2 + 2x$,
$$\begin{aligned}&2x^2 + b = -x^2 + 2x\\ \Longrightarrow\ &3x^2 - 2x + b = 0.\end{aligned}$$

We need the two parabolae to be tangent. Setting $\Delta = 0$, we want $4 - 12b = 0$, which gives the y intercept as $b = 1/3$.

4831. (a) Using the change of base formula,

$$\log_k x \equiv \frac{\log_e x}{\log_e k} \equiv \frac{\ln x}{\ln k}.$$

So, the derivative of $y = \log_k x$ is

$$\frac{dy}{dx} = \frac{1}{x \ln k}.$$

At $x = 1$, this has value $\frac{1}{\ln k}$. So, the equation of the tangent line is

$$y = \frac{x-1}{\ln k}.$$

(b) The second derivative is

$$\frac{dy}{dx} = -\frac{1}{x^2 \ln k}.$$

For $x > 0$, $x^2 > 0$. And $k > 1$, so $\ln k > 0$. Hence, the second derivative is negative for $x > 0$, signifying that the curve is concave.

(c) Since the curve is concave, it is always at or below its tangent line at $x = 1$. Hence, I is bounded by the area of the region under the tangent, for $x \in [1, k]$. This is a triangle with base $(k-1)$ and height $\frac{1}{\ln k}(k-1)$. Therefore, using $A = \frac{1}{2}bh$,

$$I < \frac{(k-1)^2}{2 \ln k}, \text{ as required.}$$

4832. (a) Differentiating implicitly,

$$\begin{aligned} & x^2 \sin y + \cos y = x \\ \Longrightarrow\ & 2x \sin y + x^2 \cos y \frac{dy}{dx} - \sin y \frac{dy}{dx} = 1 \\ \Longrightarrow\ & \frac{dy}{dx}\left(x^2 \cos y - \sin y\right) = 1 - 2x \sin y \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{1 - 2x \sin y}{x^2 \cos y - \sin y}. \end{aligned}$$

(b) For SPs, $1 - 2x \sin y = 0$. Rearranging this to $x = {}^1\!/_{2 \sin y}$, we substitute in:

$$\begin{aligned} & \frac{1}{4 \sin^2 y} \sin y + \cos y = \frac{1}{2 \sin y} \\ \Longleftrightarrow\ & \tfrac{1}{4} + \sin y \cos y = \tfrac{1}{2} \\ \Longleftrightarrow\ & 4 \sin y \cos y = 1. \end{aligned}$$

(c) Solving the above,

$$\begin{aligned} & 4 \sin y \cos y = 1 \\ \Longleftrightarrow\ & \sin 2y = \tfrac{1}{2} \\ \Longleftrightarrow\ & 2y = \tfrac{\pi}{6}, \tfrac{5\pi}{6}, ... \\ \Longleftrightarrow\ & y = \tfrac{\pi}{12}, \tfrac{5\pi}{12}, ... \end{aligned}$$

Each of these infinitely many y values produces one x value via $x = {}^1\!/_{2 \sin y}$. Hence, there are infinitely many SPs, as required.

4833. We are given that the sum of the four scores is 12. We are looking for the product to be 64, which is 2^6. So, each score X_i must be $1, 2$ or 4. There is only one combination of four of these which adds to 12: $4 + 4 + 2 + 2$. We need to find

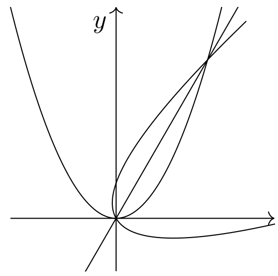
$$\mathbb{P}\left(\{4,4,2,2\} \text{ in some order} \,\middle|\, \sum X_i = 12\right).$$

We classify outcomes by largest score:

Largest	Others	Orders
3	$\{3,3,3\}$	1
4	$\{4,3,1\}$	12
4	$\{4,2,2\}$	**6**
4	$\{3,3,2\}$	12
5	$\{5,1,1\}$	6
5	$\{4,2,1\}$	24
5	$\{3,3,1\}$	12
5	$\{3,2,2\}$	12
6	$\{2,2,2\}$	4
6	$\{1,2,3\}$	24
		113.

So, $p = \frac{6}{113}$, as required.

4834. Sketch:



The area enclosed by the curves is twice the area enclosed by $y = x^2$ and $y = \sqrt{3}x$. These intersect at $x = 0$ and $x = \sqrt{3}$. So, the area is

$$\begin{aligned} A &= 2\int_0^{\sqrt{3}} \sqrt{3}x - x^2 \, dx \\ &= 2\left[\tfrac{\sqrt{3}}{2}x^2 - \tfrac{1}{3}x^3\right]_0^{\sqrt{3}} \\ &= 2\left(\tfrac{3\sqrt{3}}{2} - \sqrt{3}\right) \\ &= \sqrt{3}, \text{ as required.} \end{aligned}$$

4835. When writing an expression in harmonic form, the usual method is to take the primary solutions of both R and α. With equations $R \sin \alpha = a$ and $R \cos \theta = b$, these are

① $R = +\sqrt{a^2 + b^2}$,

② $\alpha = \arctan \frac{a}{b}$.

However, while each solution is individually valid, these values of R (as opposed to its negative) and α (as opposed to $\alpha + \pi$) only provide a solution *together* if a and b have the same sign. The usual technique is to match the signs beforehand, by choosing the correct harmonic form. To proceed as simply as possible, the student should choose $R\sin(\theta-\alpha)$. Then, her working would go:

$$\begin{aligned}&3\sin\theta - 4\cos\theta \equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha\\ &\text{Equating coeffs, } R\cos\alpha = 3 \text{ and } R\sin\alpha = 4\\ &\text{Hence, } R = 5 \text{ and } \alpha = \arctan\tfrac{4}{3}.\\ &\text{This gives } 5\sin\left(\theta - \arctan\tfrac{4}{3}\right).\end{aligned}$$

Nota Bene

It is also possible to fix the student's working in a slightly more laborious fashion, by taking e.g. the value $R = 5$ and substituting it back in to find α. This is entirely valid mathematically, but isn't quite as slick. Harmonic form is simpler if you choose the right form from the get go.

4836. Differentiating,

$$\begin{aligned}y &= 4x^4 - 4x^3 - 7x^2 + 4x + 3\\ \Longrightarrow \frac{dy}{dx} &= 16x^3 - 12x^2 - 14x + 4.\end{aligned}$$

Solving for SPs,

$$\begin{aligned}16x^3 - 12x^2 - 14x + 4 &= 0\\ \Longrightarrow x &= \tfrac{1}{4}, \tfrac{1}{4}\left(1 \pm \sqrt{17}\right).\end{aligned}$$

These are in arithmetic progression around $x = \frac{1}{4}$. Checking the y coordinates, the y coordinates of each of the outer two SPs is -1. Since the SPs are symmetrical in $x = \frac{1}{4}$, so must the curve be.

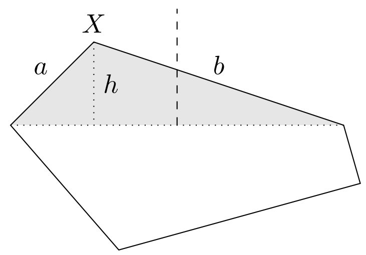
Alternative Method

Having determined that $x = \frac{1}{4}$ is the putative line of symmetry, let $z = x - \frac{1}{4}$. Substituting $x = z + \frac{1}{4}$, the equation of the curve is

$$\begin{aligned}y &= 4\left(z+\tfrac{1}{4}\right)^4 - 4\left(z+\tfrac{1}{4}\right)^3 - 7\left(z+\tfrac{1}{4}\right)^2\\ &\qquad + 4\left(z+\tfrac{1}{4}\right) + 3\\ &\equiv \tfrac{1}{64}\left(256z^4 - 544z^2 + 255\right).\end{aligned}$$

Since this is a biquadratic, it must have $z = 0$ as a line of symmetry. Hence, the original quartic has $x = \frac{1}{4}$ as a line of symmetry.

4837. Assume, for a contradiction, that there exists an irregular pentagon, with perimeter P, which has maximal area. Let a and b be the lengths of two adjacent sides, where $a \neq b$. The pentagon may be partitioned into a triangle and a quadrilateral, as follows:



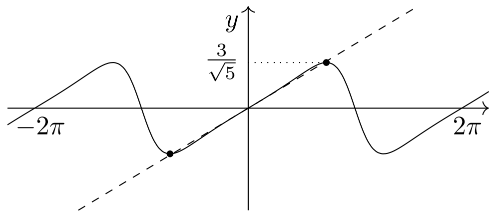
Move X onto the dashed perpendicular bisector, leaving the other vertices where they are. Since the total length $a + b$ is fixed, this must increase the perpendicular height h of the shaded triangle, thus increasing its area. Hence, the new pentagon has a greater area than the original one. This is a contradiction. Therefore, for a fixed perimeter P, the pentagon of greatest area is regular. □

4838. (a) For stationary points with $x \in [-\pi, \pi]$,

$$\begin{aligned}&3\cos x(3 + 2\cos x) - 3\sin x \cdot -2\sin x = 0\\ \Longrightarrow\ &3\cos x + 2\cos^2 x + 2\sin^2 x = 0\\ \Longrightarrow\ &3\cos x = -2\\ \Longrightarrow\ &x = \pm\arccos\left(-\tfrac{2}{3}\right)\\ &\quad = \pm\left(\pi - \arccos\tfrac{2}{3}\right)\\ &\quad = \pm\pi \mp \arccos\tfrac{2}{3}.\end{aligned}$$

(b) At $x = 0$, the tangent line is $y = 0.6x$. Since the next five derivatives are all close to zero, the gradient deviates very little from 0.6 in a reasonably large domain around $x = 0$. Hence, the curve is very well approximated by $y = 0.6x$ for most of the domain between the SPs in part (a).

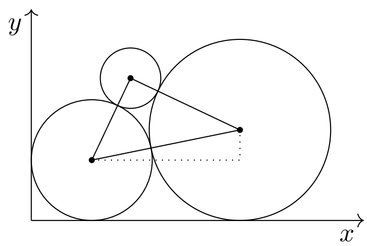
(c) The curve is periodic, with period 2π. So, the behaviour described in part (b) is repeated around $x = 2n\pi$ for all $n \in \mathbb{Z}$:



4839. The minimal width of the rectangle is 6. It is possible to fit the circles into a 6×10 rectangle. The rectangle required is slightly smaller than this, with area

$$30 + 12\sqrt{6} \approx 59.4.$$

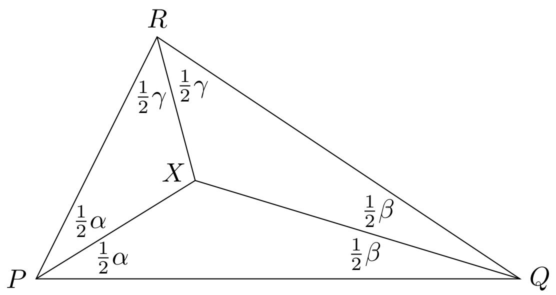
Rotate the picture so that the two larger circles are tangent to the x axis:



The centres form a $(3, 4, 5)$ triangle. The dotted triangle has a hypotenuse of 5 and $\Delta y = 1$, so $\Delta x = \sqrt{24} = 2\sqrt{6}$. So, the collective x width of the circles is $5 + 2\sqrt{6}$. Hence, they will fit inside a rectangle, sides parallel to the axes, of area

$$\left(5 + 2\sqrt{6}\right) \times 6 = 30 + 12\sqrt{6}, \text{ as required.}$$

4840. The string is smooth, so, by symmetry, the lines of action must lie along the angle bisectors of the triangle of string, meeting at the incentre X:



Angle RXQ is $180° - \frac{1}{2}\beta - \frac{1}{2}\gamma$. Since the lines of action meet at X, this is the obtuse angle between the forces at R and Q meeting tip-to-tip. So, the acute angle between the forces at R and Q meeting tip-to-tail is

$$\begin{aligned} A &= 180° - \angle RXQ \\ &= 180° - \left(180° - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma\right) \\ &= \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma. \end{aligned}$$

Since $\frac{1}{2}(\alpha + \beta + \gamma) = 90°$, this is

$$A = 90° - \tfrac{1}{2}\alpha.$$

The other results follow by symmetry.

4841. Let Y be the integral of y. Then we can rewrite the equation as a DE, absorbing the $+c$ into Y:

$$\begin{aligned} & Y = \sqrt{1 - \left(\frac{dY}{dx}\right)^2} \\ \Longrightarrow\ & Y^2 = 1 - \left(\frac{dY}{dx}\right)^2 \\ \Longrightarrow\ & \frac{dY}{dx} = \pm\sqrt{1 - Y^2}. \end{aligned}$$

Separating the variables,

$$\int \frac{1}{\sqrt{1 - Y^2}}\, dY = \pm \int 1\, dx.$$

The RHS is $\pm x + c$. To integrate the LHS, let $Y = \sin\theta$, so that $dY = \cos\theta\, d\theta$. Enacting the substitution,

$$\begin{aligned} & \int \frac{1}{\sqrt{1 - Y^2}}\, dY \\ &= \int \frac{1}{\sqrt{1 - \sin^2\theta}} \cos\theta\, d\theta \\ &= \int 1\, d\theta \\ &= \theta + d \\ &= \arcsin Y + d. \end{aligned}$$

In the original equation, we combine the constants of integration:

$$\begin{aligned} & \arcsin Y = \pm x + k \\ \Longrightarrow\ & Y = \sin(\pm x + k). \end{aligned}$$

Differentiating, $y = \pm\cos(\pm x + k)$. The $\pm$ signs are made redundant by the presence of k. So, in its simplest form, the general solution is

$$y = \cos(x + k).$$

— Nota Bene —

This general solution could also be written

$$y = \sin(x + k).$$

Had we used the substitution $x = \sin\theta$, this would have emerged from the algebra. The solutions are equivalent, as sin and cos waves are translations of one another.

4842. Edge V_1V_2 has two endpoints, from each of which there are two edges besides V_1V_2. Having chosen V_1V_2 wlog, there are five edges which cannot be chosen as V_3V_4. Seven edges remain in the initial possibility space.

Any rhombus congruent to the one depicted has midpoints on parallel edges which are diagonally opposite each other. Of the seven edges remaining to be chosen, only one is parallel to V_1V_2 and also diagonally opposite it. So, the probability that V_3V_4 is chosen successfully is $\frac{1}{7}$.

With V_1V_2 and V_3V_4 chosen, four vertices remain. There are ${}^4\mathrm{C}_2 = 6$ ways of choosing two of these. Two of these ways produce a rhombus. So, the overall probability of ending up with a rhombus congruent to the one depicted is

$$p = \tfrac{1}{7} \times \tfrac{2}{6} = \tfrac{1}{21}.$$

4843. The curves are a circle, centre $(0, 6)$, radius 4, and a parabola. The shortest distance between them lies along the normal to both. This is an extended radius of the circle, and must pass through $(0, 6)$.

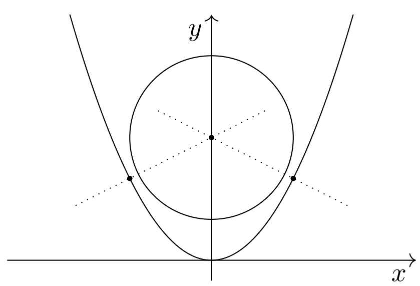
The equation of a generic normal at $x = p$ is

$$y - \tfrac{1}{4}p^2 = -\tfrac{2}{p}(x - p).$$

Substituting $(0, 6)$ into this,

$$\begin{aligned} & 6 - \tfrac{1}{4}p^2 = 2 \\ \Longrightarrow\ & p = \pm 4. \end{aligned}$$

The relevant points on the curve are $(4, 4)$ and $(-4, 4)$. For both, the distance to $(0, 6)$ is $2\sqrt{5}$. Subtracting the radius, the shortest distance is $2\sqrt{5} - 4$.



4844. The form for the partial fractions is

$$\frac{1}{x^2(x+1)^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

Multiplying up by the denominators,

$$1 \equiv Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2.$$

Substituting $x = 0$ gives $B = 1$ and $x = -1$ gives $D = 1$. Then, equating coefficients,

$$\begin{aligned} x^3 &: 0 = A + C, \\ x^2 &: 0 = 2A + 1 + C + 1. \end{aligned}$$

Solving these, $A = -2$ and $C = 2$. The indefinite integral is

$$\begin{aligned} &\int \frac{1}{x^2(x+1)^2}\, dx \\ &= \int -\frac{2}{x} + \frac{1}{x^2} + \frac{2}{x+1} + \frac{1}{(x+1)^2}\, dx \\ &= -2\ln|x| - x^{-1} + 2\ln|x+1| - (x+1)^{-1} + c \\ &\equiv 2\ln\left|\frac{x+1}{x}\right| - x^{-1} - (x+1)^{-1} + c. \end{aligned}$$

So, the limit is

$$\lim_{L\to\infty} \left[2\ln\left|\frac{x+1}{x}\right| - x^{-1} - (x+1)^{-1} \right]_{\frac{1}{2}}^{L}.$$

Evaluating at L, and then taking the limit $L \to \infty$, yields zero, because the input of the ln function tends to 1 and both other terms tend to zero. So, the value of the entire integral is

$$\begin{aligned} &-\left(2\ln\left|\frac{\frac{1}{2}+1}{\frac{1}{2}}\right| - \left(\tfrac{1}{2}\right)^{-1} - \left(\tfrac{1}{2}+1\right)^{-1} \right) \\ &= -\left(2\ln 3 - 2 - \tfrac{2}{3}\right) \\ &= \tfrac{8}{3} - 2\ln 3. \end{aligned}$$

4845. Firstly, consider $y = \mathrm{gh}(x) = 2x^3 - 6x + 1$. This is a cubic with three distinct x intercepts, as can be seen from locating SPs:

$$\begin{aligned} 6x^2 - 6 &= 0 \\ \Longrightarrow x &= \pm 1. \end{aligned}$$

The SPs are $(-1, 5)$, $(1, -3)$. The y coordinates are positive and negative, so the cubic has three distinct x intercepts, which are single roots.

Applying f to $y = \mathrm{gh}(x)$, each of the factors is squared. So, the curve is now a sextic with three double roots. Each of these is a stationary point on the x axis. In between these roots, there must be two SPs which are not on the x axis. And, since a sextic can never have more than five SPs, we know we have found the full complement. So, $y = \mathrm{fgh}(x)$ has five SPs, three of which lie on the x axis.

4846. The pegs are smooth, so the tension is the same throughout. Hence, the resultant force applied by the string at A acts along the angle bisector at A. Let θ be half the angle at A. Using the cosine rule,

$$\cos 2\theta = \frac{b^2 + c^2 - a^2}{2bc}$$

The resultant force exerted by the string is $2T\cos\theta$. Using a double-angle formula

$$\begin{aligned} 2\cos^2\theta - 1 &= \frac{b^2 + c^2 - a^2}{2bc} \\ \Longrightarrow 2\cos^2\theta &= \frac{b^2 + c^2 - a^2}{2bc} + 1 \\ &\equiv \frac{b^2 + 2bc + c^2 - a^2}{2bc} \\ &\equiv \frac{(b+c)^2 - a^2}{2bc}. \end{aligned}$$

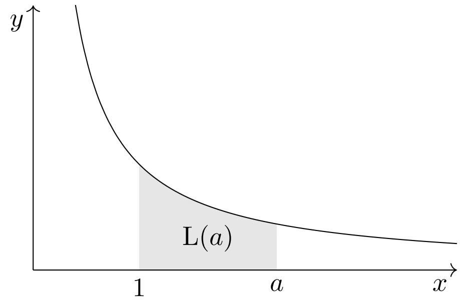
The triangle is acute, so we can take the positive square root. This gives

$$\cos\theta = \sqrt{\frac{(b+c)^2 - a^2}{4bc}}.$$

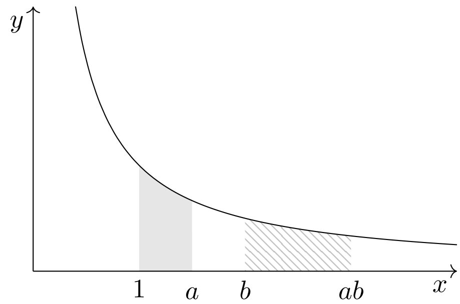
So, the force exerted at A is

$$\begin{aligned} F_A &= \sqrt{\frac{(b+c)^2 - a^2}{4bc}} \cdot 2T \\ &\equiv \sqrt{\frac{(b+c)^2 - a^2}{bc}}\, T, \text{ as required.} \end{aligned}$$

4847. (a) The relevant region is



(b) Consider the area between $x = b$ and $x = ab$:



The solid shaded region can be transformed to the hatched region by

① a stretch factor b in the x direction,
② a stretch factor $\frac{1}{b}$ in the y direction.

To prove this, consider the point $(p, 1/p)$. It is transformed first to $(bp, 1/p)$, then to $(bp, 1/bp)$, hence remaining on the curve $y = 1/x$. That the endpoints match can be easily verified.

Under the action of these two stretches, the area is unchanged. Equating the areas,

$$\begin{aligned} & \mathrm{L}(ab) - \mathrm{L}(b) = \mathrm{L}(a) \\ \Longrightarrow\ & \mathrm{L}(a) + \mathrm{L}(b) = \mathrm{L}(ab), \text{ as required.} \end{aligned}$$

4848. Differentiating $y = ux$,

$$\frac{dy}{dx} = \frac{du}{dx}x + u.$$

Enacting the substitution,

$$\begin{aligned} & \frac{du}{dx}x + u = \frac{u^2x^2 - ux^2}{x^2} \\ \Longrightarrow\ & \frac{du}{dx}x = u^2 - 2u. \end{aligned}$$

Separating the variables,

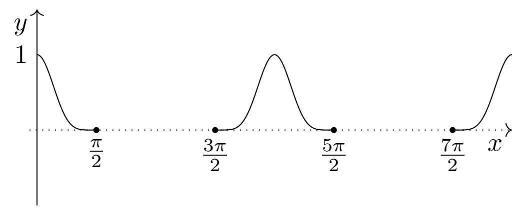
$$\begin{aligned} & \int \frac{1}{u(u-2)}\, du = \int \frac{1}{x}\, dx \\ \Longrightarrow\ & \tfrac{1}{2}\int \frac{1}{u-2} - \frac{1}{u}\, du = \ln|x| \\ \Longrightarrow\ & \ln\left|\frac{u-2}{u}\right| = 2\ln|x| + c \\ \Longrightarrow\ & \ln\left|\frac{u-2}{u}\right| = \ln(x^2) + c \\ \Longrightarrow\ & \frac{u-2}{u} = Ax^2. \end{aligned}$$

This gives

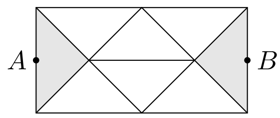
$$\begin{aligned} & \frac{\frac{y}{x} - 2}{\frac{y}{x}} = Ax^2 \\ \Longrightarrow\ & y - 2x = Ax^2y \\ \Longrightarrow\ & y(1 - Ax^2) = 2x \\ \Longrightarrow\ & y = \frac{2x}{1 - Ax^2}. \end{aligned}$$

4849. The index 5.5 is $11/2$. The numerator is odd, which maintains signs. The denominator is even, so the curve has no points where $\cos x < 0$.

In raising to the power 5.5, the x intercepts of $y = \cos x$ are flattened greatly, as they are changed from single roots to 5.5-tuple roots. So, we have an interrupted oscillation between $y = 0$ and $y = 1$, with flat turning points at $y = 0$ and tight turning points at $y = 1$. This is shown below; the x axis is dotted to bring out the behaviour.



4850. The regions containing A and B must be shaded. So, with probability $\frac{1}{4}$, we start with



From here, there are $2^6 = 64$ outcomes. Classify the successful ones by number n of shaded regions. Clearly $n = 0, 1, 2$ are unsuccessful.

③ There are 2 successful outcomes: three across the top or three across the bottom.

④ There are two options:
- the top row or bottom row forms a path, with one extra, giving 6 outcomes,
- four are shaded as in the example in the question, giving 2 outcomes.

⑤ All 6 outcomes are successful.

⑥ The 1 outcome is successful.

Adding these up, the total number of successful outcomes is $2 + (6 + 2) + 6 + 1 = 17$. This gives the probability as

$$p = \tfrac{1}{4} \times \tfrac{17}{64} = \tfrac{17}{256}, \text{ as required.}$$

4851. The mass is 1, so the force gives the acceleration directly. The acceleration is 1 ms^{-2} for 1 second, then 2 ms^{-2} for 1 second, and so on, until it is n ms^{-2} for 1 second. Thereafter, the object moves at constant velocity.

(a) The total change in velocity is

$$\Delta v = 1 + 2 + 3 + ... + n = \tfrac{1}{2}n(n+1).$$

After $t = n$, there is no change in velocity. So, at $t = 2n$, the velocity is $\frac{1}{2}n(n+1)$ ms^{-1}.

(b) The displacement is given by the area under a velocity-time graph. The velocity is

$$\begin{aligned} & \tfrac{1}{2} \cdot 1 \cdot (1+1) = 1 \text{ for 1 second,} \\ & \tfrac{1}{2} \cdot 2 \cdot (2+1) = 3 \text{ for 1 second,} \\ & \tfrac{1}{2} \cdot 3 \cdot (3+1) = 6 \text{ for 1 second,} \\ & ... \\ & \tfrac{1}{2}(n-1)n \text{ for 1 second,} \\ & \tfrac{1}{2}n(n+1) \text{ for } n+1 \text{ seconds.} \end{aligned}$$

For $t \in [0, n)$, the relevant sum is

$$\begin{aligned} & 1 + 3 + 6 + ... + \tfrac{1}{2}n(n+1) \\ \equiv\ & \sum_{r=1}^{n} \tfrac{1}{2}r(r+1) \\ \equiv\ & \tfrac{1}{2}\sum_{r=1}^{n} r^2 + \tfrac{1}{2}\sum_{r=1}^{n} r \\ \equiv\ & \tfrac{1}{12}n(n+1)(2n+1) + \tfrac{1}{4}n(n+1) \\ \equiv\ & \tfrac{1}{6}n(n+1)(n+2). \end{aligned}$$

After this, the object moves at $\frac{1}{2}n(n+1)$ for n seconds, travelling a distance of $\frac{1}{2}n^2(n+1)$. So, the total displacement at $t = 2n$ is

$$\begin{aligned} s &= \tfrac{1}{6}n(n+1)(n+2) + \tfrac{1}{2}n^2(n+1) \\ &\equiv \tfrac{1}{3}n(n+1)(2n+1) \text{ m.} \end{aligned}$$

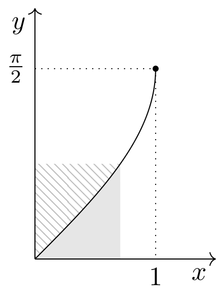
4852. There is a common factor of $(x-1)$ in the top and bottom, which needs to come out before we can take the limit:

$$\begin{aligned} &\lim_{x\to 1} \frac{x^n - 1}{x^n - x} \\ =\ &\lim_{x\to 1} \frac{(x-1)(x^{n-1} + x^{n-2} + ... + x + 1)}{(x-1)(x^{n-1} + x^{n-2} + ... + x)} \\ =\ &\lim_{x\to 1} \frac{x^{n-1} + x^{n-2} + ... + x + 1}{x^{n-1} + x^{n-2} + ... + x}. \end{aligned}$$

At this point, we can take the limit. Each term in the numerator and denominator tends to 1. Counting up the number of terms in each case,

$$\begin{aligned} &\lim_{x\to 1} \frac{x^n - 1}{x^n - x} \\ \equiv\ &\frac{n}{n-1}. \end{aligned}$$

4853. The domain of the arcsin function is $[-1, 1]$. It has odd symmetry, so we need address only $[0, 1]$. Consider the indefinite integral as an area function up to x:



The integral of arcsin is the solid shaded area. We can express this as the rectangle minus the hatched area. This is

$$\int_0^x \arcsin s \, ds = x \arcsin x - \int_0^y \sin t \, dt.$$

When we integrate, the constant in the y integral can be absorbed into a generic $+c$:

$$\begin{aligned} \int \arcsin x \, dx &= x \arcsin x + \cos y + c \\ &= x \arcsin x + \sqrt{1 - \sin^2 y} + c \\ &= x \arcsin x + \sqrt{1 - x^2} + c. \end{aligned}$$

— Nota Bene —

This result can also be shown using algebraically using integration by parts.

4854. We require a and $a+2$ both to be prime. So, the number $a+1$ between the primes must be even. Consider whether $a+1$ is also a multiple of 3:

① If $a+1$ is not a multiple of three, then one of a or $a+2$ is. But they are prime. So, one must be 3 itself. Since 1 is not prime, this gives the first twin prime pair $(3, 5)$.

② If $a+1$ is a multiple of three, then, since it is even, it is a multiple of 6 and can be written $6k$, for $k \in \mathbb{N}$. The sum of the twin primes is

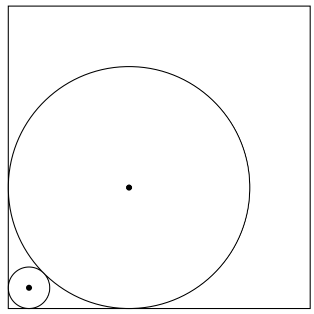
$$\begin{aligned} &(6k-1) + (6k+1) \\ \equiv\ &12k. \end{aligned}$$

This is divisible by 12.

Quod erat demonstrandum.

4855. There are three cases to consider, each of which produces different algebra:

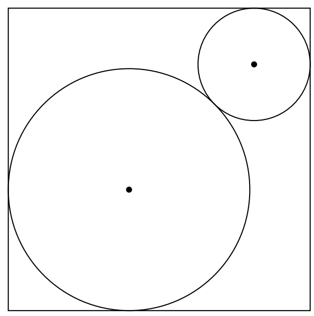
① Suppose the circles are tangent to the same two sides of the square. This scenario is



In this case, the radii are related by

$$\begin{aligned} \sqrt{2}r + r + R &= \sqrt{2}R \\ \Longrightarrow r &= \frac{\sqrt{2}-1}{\sqrt{2}+1}R \\ &\equiv (3 - 2\sqrt{2})R. \end{aligned}$$

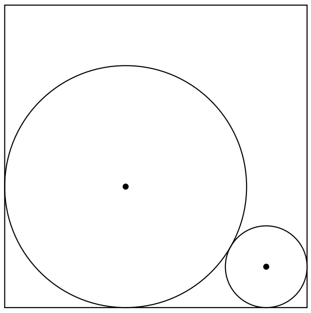
② Suppose, alternatively, that the circles are tangent to opposing pairs of sides. This is



In this case, the radii are related by

$$\begin{aligned} (\sqrt{2}+1)r + (\sqrt{2}+1)R &= \sqrt{2} \\ \Longrightarrow r &= \frac{\sqrt{2}}{\sqrt{2}+1} - R \\ &\equiv 2 - \sqrt{2} - R. \end{aligned}$$

③ Suppose, alternatively, that the circles have one common side as a tangent. This is



In this case, the radii are related by

$$\begin{aligned} & R+\sqrt{(R+r)^2-(R-r)^2}+r=1 \\ \Longrightarrow\ & R+2\sqrt{Rr}+r=1 \\ \Longrightarrow\ & \left(\sqrt{R}+\sqrt{r}\right)^2=1. \end{aligned}$$

Taking the positive square root,

$$\begin{aligned} & \sqrt{r}=1-\sqrt{R} \\ \Longrightarrow\ & r=\left(1-\sqrt{R}\right)^2. \end{aligned}$$

So, as required, one of the following holds:

$$r=\begin{cases} \left(3-2\sqrt{2}\right)R, \\ \dfrac{2-2R}{1+\sqrt{2}}, \\ \left(1-\sqrt{R}\right)^2. \end{cases}$$

4856. Differentiating $\mathrm{E}(x)$,

$$\begin{aligned} & \mathrm{E}(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+... \\ \Longrightarrow\ & \mathrm{E}'(x)=0+1+\frac{2x}{2!}+\frac{3x^2}{3!}+\frac{3x^2}{3!}+... \\ & \equiv 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+... \\ & =\mathrm{E}(x). \end{aligned}$$

Let $y=\mathrm{E}(x)$. Then, the above is $\frac{dy}{dx}=y$. This is a separable DE. Separating the variables,

$$\begin{aligned} & \int \frac{1}{y}\,dy=\int 1\,dx \\ \Longrightarrow\ & \ln|y|=x+c \\ \therefore\ & y=Ae^x, \text{ for some constant } A. \end{aligned}$$

Substituting $x=0$, we get $\mathrm{E}(x)=y=1$, so $A=1$. Hence, $y=\mathrm{E}(x)=e^x$, proving (on the assumption of convergence) the following series expansion:

$$e^x\equiv 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...$$

4857. Substituting the parametric equations in, the LHS of the equation of the ellipse is

$$\begin{aligned} & 8\sin^2 2t+4\sin^2 t \\ \equiv\ & 32\sin^2 t\cos^2 t+4\sin^2 t \\ \equiv\ & 32\sin^2 t(1-\sin^2 t)+4\sin^2 t \\ \equiv\ & 36\sin^2 t-32\sin^4 t. \end{aligned}$$

We need to show that this can exceed 9 in value. Looking for a maximum,

$$\begin{aligned} & 72\sin t\cos t-128\sin^3 t\cos t=0 \\ \Longrightarrow\ & \cos t(9-16\sin^2 t)=0. \end{aligned}$$

Setting aside $\cos t=0$, we have

$$\sin^2 t=\tfrac{9}{16}.$$

So, consider the point at $t=\arcsin\frac{3}{4}$. At this point, the coordinates are

$$\left(\tfrac{3\sqrt{7}}{8},\tfrac{3}{4}\right).$$

Evaluating at this point,

$$8x^2+4y^2\,\Big|_{\left(\frac{3\sqrt{7}}{8},\frac{3}{4}\right)}=\tfrac{81}{8}>9.$$

So, this point and therefore part of the curve lies outside the ellipse $8x^2+4y^2=9$.

4858. Let the sequences be

$$\begin{aligned} \{u_n\} &= \{a, a+d, a+2d, ...\} \\ \{v_n\} &= \{a, ar, ar^2, ...\} \end{aligned}$$

We are told that the third terms are equal, so that

$$a+2d=ar^2.$$

Also, the second terms differ by one, so that

$$a+d-ar=\pm 1.$$

Substituting for d,

$$\begin{aligned} & a+2(\pm 1+ar-a)=ar^2 \\ \Longrightarrow\ & \pm 2+2ar-a=ar^2 \\ \Longrightarrow\ & \pm\tfrac{2}{a}=r^2-2r+1 \\ & \equiv (r-1)^2 \end{aligned}$$

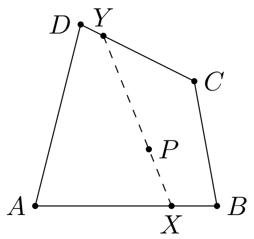
Since $r\in\mathbb{Z}$, $(r-1)^2\in\mathbb{Z}$. So, $\frac{2}{a}\in\mathbb{Z}$. And a is also an integer. The only possibilities are therefore $a=1,2$. The former gives $r-1=\pm\sqrt{2}$, which is not an integer. So, $a=2$ is the only possibility. With $a=2$, $r-1=\pm 1$, so $r=0,2$. This gives two possibilities:

$\{u_n\}$	$\{2,1,0,...\}$	$\{2,5,8,...\}$
$\{v_n\}$	$\{2,0,0,...\}$	$\{2,4,8,...\}$

4859. Let points X and Y have position vectors

$$\overrightarrow{OX} = \frac{\lambda_1\mathbf{a} + \lambda_2\mathbf{b}}{\lambda_1 + \lambda_2},$$
$$\overrightarrow{OY} = \frac{\lambda_3\mathbf{c} + \lambda_4\mathbf{d}}{\lambda_3 + \lambda_4}.$$

According to the assumed result, we know that X lies on AB and Y lies on CD.



We can now define point P by

$$\begin{aligned}\overrightarrow{OP} &= \lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c} + \lambda_4\mathbf{d}\\ &= (\lambda_1 + \lambda_2)\overrightarrow{OX} + (\lambda_3 + \lambda_4)\overrightarrow{OY}.\end{aligned}$$

This tells us that P lies on the chord XY. Since $ABCD$ is convex, XY lies within it. So, point P lies within the quadrilateral $ABCD$. □

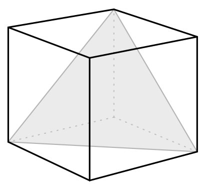
4860. Rearranging $y = \arctan(\cos x)$ gives $\tan y = \cos x$. Differentiating implicitly, then using the second Pythagorean identity,

$$\begin{aligned}&\sec^2 y\frac{dy}{dx} = -\sin x\\ \implies &(1 + \tan^2 y)\frac{dy}{dx} = -\sin x.\end{aligned}$$

Substituting $\tan y = \cos x$,

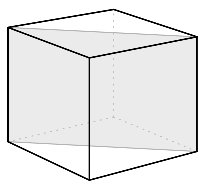
$$\begin{aligned}&(1 + \cos^2 x)\frac{dy}{dx} = -\sin x\\ \implies &\frac{dy}{dx} = \frac{-\sin x}{1 + \cos^2 x}, \text{ as required.}\end{aligned}$$

4861. (a) In a triangular cross-section, the greatest side length is $\sqrt{2}$. The following is maximal:



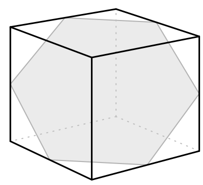
The area is $\frac{\sqrt{3}}{2} \approx 0.87$.

(b) With a quadrilateral, we can't make all sides $\sqrt{2}$. So, the greatest area isn't the symmetrical square. A rectangle is maximal:



The area is $\sqrt{2} \approx 1.4$.

(c) Symmetry dictates that the hexagonal cross-section of greatest area must be regular. In this hexagon, every side joins two adjacent midpoints, with length $\sqrt{2}/2$.



The area of this hexagon is $\frac{3\sqrt{3}}{4} \approx 1.3$

— Nota Bene —

In a way, it is surprising that the cross-section with greatest area is not the symmetrical hexagon, but is rather the asymmetrical rectangle. However, it is less surprising when one considers that the rectangle and the cube share angles, whereas the hexagon and the cube do not.

4862. (a) ① $\Longleftarrow$ ②. If the range of f'' is $\mathbb{R}$, then f'' is a polynomial of odd degree. (A polynomial of even degree has a global min or max.)

Integrating twice, f is then also a polynomial of odd degree, so has range $\mathbb{R}$. Hence, if the range of f'' is $\mathbb{R}$, then the range of f is $\mathbb{R}$. QED.

(b) Consider $f(x) = x$. This is a linear polynomial of odd degree (1), with range $\mathbb{R}$. Its second derivative is $f''(x) = 0$, which has range $\{0\}$. This is a counterexample to the forwards and therefore the two-way implication.

4863. Dividing top and bottom by x^2 and letting $x \to \infty$, the equation of the horizontal asymptote is $y = 2$. So, the centre of the circle is $(0, 2)$.

The gradient of the curve is

$$\begin{aligned}\frac{dy}{dx} &= \frac{4x(x^2+1) - 2x^2(2x)}{\left(x^2+1\right)^2}\\ &\equiv \frac{4x}{\left(x^2+1\right)^2}.\end{aligned}$$

So, the equation of the normal at $x = a$ is

$$y - \frac{2a^2}{a^2+1} = -\frac{\left(a^2+1\right)^2}{4a}(x - a).$$

We need this to pass through $(0, 2)$. Subbing in,

$$\begin{aligned}&2 - \frac{2a^2}{a^2+1} = -\frac{\left(a^2+1\right)^2}{4a}(-a)\\ \implies &\frac{2}{a^2+1} = \frac{\left(a^2+1\right)^2}{4}\\ \implies &8 = (a^2+1)^3\\ \implies &a^2 + 1 = 2\\ \implies &a = \pm 1.\end{aligned}$$

This gives the points of tangency as $(\pm 1, 1)$, so the radius of the circle is $r = \sqrt{2}$.

4864. We can rewrite the integrand as

$$\frac{a^2+x^2}{b^2+x^2} \equiv 1 + \frac{a^2-b^2}{b^2+x^2}.$$

Let $x = b\tan\theta$, so that $dx = b\sec^2\theta\, d\theta$. This gives the indefinite integral as

$$\begin{aligned}
&\int 1 + \frac{a^2-b^2}{b^2+x^2}\, dx \\
&= \int \left(1 + \frac{a^2-b^2}{b^2+b^2\tan^2\theta}\right) b\sec^2\theta\, d\theta \\
&= \int b\sec^2\theta + \frac{a^2-b^2}{b}\, d\theta \\
&= b\tan\theta + \frac{(a^2-b^2)\theta}{b} + c \\
&= x + \frac{(a^2-b^2)\arctan\frac{x}{b}}{b} + c.
\end{aligned}$$

So, the definite integral from $x = 0$ to $x = b$ is

$$\begin{aligned}
&b + \frac{(a^2-b^2)\arctan 1}{b} - (0+0) \\
\equiv\ & b + \frac{\pi(a^2-b^2)}{4b}, \text{ as required.}
\end{aligned}$$

4865. For small angles in radians, $y = \cos x$ is close to $(0, 1)$. So, the circular arc in question must be the unit circle centred on the origin. Its upper half has equation

$$y_{\text{arc}} = (1-x^2)^{-\frac{1}{2}}.$$

We expand binomially. Since x is small, we ignore terms in x^4 and above. This gives

$$\begin{aligned}
y_{\text{arc}} &= 1 - \left(-\tfrac{1}{2}\right)(-x^2) + \ldots \\
&\equiv 1 - \tfrac{1}{2}x^2 + \ldots
\end{aligned}$$

This is precisely the small-angle approximation for $\cos x$. Hence, $y = \cos x$ is well approximated, for small angles in radians, by an arc of the unit circle.

— Alternative Method —

We compare values of the zeroth, first and second derivatives, evaluated at the origin. Let

$$\mathrm{C}(x) = (1-x^2)^{-\frac{1}{2}}.$$

The derivatives are

$$\begin{aligned}
\mathrm{C}'(x) &= -x(1-x^2)^{-\frac{1}{2}}, \\
\mathrm{C}''(x) &= -(1-x^2)^{-\frac{3}{2}}.
\end{aligned}$$

Evaluating the functions and their derivatives,

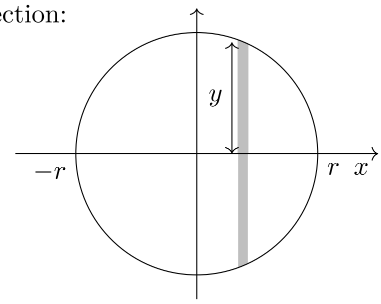
$\cos(0) = 1$	$\mathrm{C}(0) = (1-0^2) = 1$
$-\sin(0) = 0$	$\mathrm{C}'(0) = -0(1-0^2)^{-\frac{1}{2}} = 0$
$-\cos(0) = -1$	$\mathrm{C}''(0) = -(1-0^2)^{-\frac{3}{2}} = -1.$

The functions match up to the second derivative, so, in the vicinity of $x = 0$, $\cos(x) \approx \mathrm{C}(x)$.

4866. The nine points form a regular nonagon. At the centre of the circle, one edge subtends 40° and has length $2\sin 20°$. Each chord, which covers four edges, subtends 160° and has length $2\sin 80°$. So, the shaded triangle has side length

$$\begin{aligned}
&2\sin 80° - 4\sin 20° \\
&= 2\cos 10° - 8\sin 10°\cos 10° \\
&= 2\cos 10°(1 - 4\sin 10°), \text{ as required.}
\end{aligned}$$

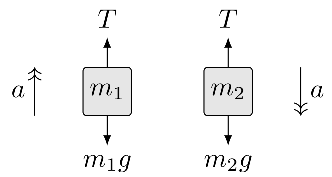
4867. Cross-section:



The circle has equation $y^2 = r^2 - x^2$. So, the shaded strip shown can be seen as the cross-section of (approximately) a disc of area $y^2 = \pi(r^2 - x^2)$. Hence, the volume of the sphere is the sum of all such discs, taken in the limit as the discs become infinitely thin. This is a definite integral:

$$\begin{aligned}
V_{\text{sphere}} &= \int_{-r}^{r} \pi(r^2 - x^2)\, dx \\
&\equiv \Big[r^2x + \tfrac{1}{3}r^3\Big]_{-r}^{r} \\
&\equiv \left(\tfrac{2}{3}\pi r^3\right) - \left(-\tfrac{2}{3}\pi r^3\right) \\
&\equiv \tfrac{4}{3}\pi r^3, \text{ as required.}
\end{aligned}$$

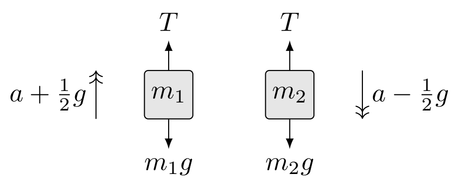
4868. (a) If the pulley does not accelerate, then it acts like a fixed pulley. The force diagrams are



The equation of motion along the string is

$$\begin{aligned}
m_2g - m_1g &= (m_1+m_2)a \\
\Longrightarrow a &= \frac{(m_2-m_1)g}{m_1+m_2}.
\end{aligned}$$

(b) If the pulley accelerates upwards at $\frac{1}{2}g$, then the accelerations of the bobs now depend on both a (acceleration of the string around the pulley), and the $\frac{1}{2}g$ acceleration of the pulley:



We can no longer take NII for the system as a whole, as the individual accelerations are now different. The equations of motion are

$$\begin{aligned} T - m_1 g &= m_1(a + \tfrac{1}{2}g), \\ m_2 g - T &= m_2(a - \tfrac{1}{2}g). \end{aligned}$$

Adding these,

$$\begin{aligned} &(m_2 - m_1)g = (m_1 + m_2)a + \tfrac{1}{2}(m_1 - m_2)g \\ \Longrightarrow\ & \left(\tfrac{3}{2}m_2 - \tfrac{3}{2}m_1\right)g = (m_1 + m_2)a \\ \Longrightarrow\ & a = \frac{3(m_2 - m_1)g}{2(m_1 + m_2)}. \end{aligned}$$

4869. (a) The squared distance $|F_1P|^2$ is given by

$$\begin{aligned} &\left(a\cos t + \sqrt{a^2 - b^2}\right)^2 + b^2\sin^2 t \\ \equiv\ & a^2\cos^2 t + 2a\cos t\sqrt{a^2 - b^2} + a^2 - b^2 \\ &\qquad + b^2 \sin^2 t \\ \equiv\ & a^2 + 2a\cos t\sqrt{a^2 - b^2} + \left(a^2 - b^2\right)\cos^2 t \\ \equiv\ & \left(a + \cos t\sqrt{a^2 - b^2}\right)^2. \end{aligned}$$

So, the distance is

$$|F_1P| = a + \cos t\sqrt{a^2 - b^2}.$$

(b) Following the same argument, $|F_2P|^2$ is

$$\begin{aligned} &\left(a\cos t - \sqrt{a^2 - b^2}\right)^2 + b^2\sin^2 t \\ \equiv\ & (a - \cos t\sqrt{a^2 - b^2})^2. \end{aligned}$$

So, the distance is

$$|F_1P| = a - \cos t\sqrt{a^2 - b^2}.$$

(c) Adding the distances,

$$\begin{aligned} &|F_1P| + |F_2P| \\ =\ & \left(a + \cos t\sqrt{a^2 - b^2}\right) + \left(a - \cos t\sqrt{a^2 - b^2}\right) \\ =\ & 2a. \end{aligned}$$

This is independent of t, as required.

— Nota Bene —

The above result is what allows one of the two standard methods of drawing an ellipse, which is as follows:

① cut a piece of string (to length $|F_1P| + |F_2P|$),
② attach its two ends to two points (the foci),
③ using a pencil to keep the string taut, draw around the two foci.

The other method uses a pair of compasses and a cylindrical tin. The proof is a bit heavier!

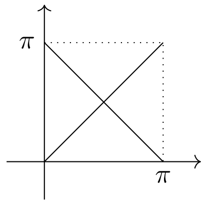
4870. Consider a network with n users. Assume, for a contradiction, that no two users have the same number of connections. The smallest possible number of connections is 0 and the greatest is $n-1$. So, there are n possible numbers of connections:

$$\underbrace{\{0, 1, 2, ..., n-1\}}_{n \text{ numbers}}.$$

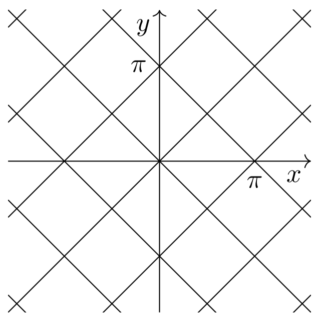
There are n users. Since no two have the same number of connections, there must be exactly one user with 0 connections, one with 1, and so on.

Consider user A, who has 0 connections, and user B, who has $n-1$ connections. A isn't connected to anybody; B is connected to everybody, which includes A. This is a contradiction. So, two users must have the same number of connections. □

4871. Since $|\sin x|$ and $|\sin y|$ are periodic, period π, we need only consider the square domain $[0, \pi] \times [0, \pi]$. On this domain, $\sin x$ and $\sin y$ are both non-negative, so the equation is $\sin x = \sin y$. This is satisfied if $y = x$ or if $y = \pi - x$. These are straight lines:



This pattern repeats periodically in both x and y. So, the entire plane is tiled with squares:



The squares have side length $\frac{\sqrt{2}}{2}\pi$, so area $\frac{1}{2}\pi^2$.

4872. (a) For SPs,

$$\begin{aligned} & -3\varepsilon x^2 + 2x = 0 \\ \Longrightarrow\ & x(-3\varepsilon x + 2) = 0 \\ \Longrightarrow\ & x = 0, \tfrac{2}{3\varepsilon}. \end{aligned}$$

Substituting these values back in, $x = 0$ gives $y = 1$ and $x = \frac{2}{3\varepsilon}$ gives

$$\begin{aligned} y &= -\varepsilon\left(\tfrac{2}{3\varepsilon}\right)^3 + \left(\tfrac{2}{3\varepsilon}\right)^2 + 1 \\ &=\equiv -\tfrac{8}{27\varepsilon^2} + \tfrac{4}{9\varepsilon^2} + 1 \\ &\equiv \tfrac{4}{27\varepsilon^2} + 1. \end{aligned}$$

So, there are two SPs, whose coordinates are

$$(0, 1) \text{ and } \left(\tfrac{2}{3\varepsilon}, \tfrac{4}{27\varepsilon^2} + 1\right).$$

(b) Consider the equation

$$-\varepsilon x^3 + x^2 + 1 = 0.$$

This is a cubic, and must have at least one real root. When x is large, comparable in size to $1/\varepsilon$, the constant term is negligible compared to the other two terms. In this regime,

$$\begin{aligned} &-\varepsilon x^3 + x^2 = 0 \\ \Longrightarrow\ & x^2(-\varepsilon x + 1) = 0. \end{aligned}$$

We reject $x = 0$, which doesn't correspond to the large-x regime. This leaves $-\varepsilon x + 1 = 0$, which is $x = 1/\varepsilon$. This *is* in the large-x regime, so is an approximate root. And there can be no further roots, since the y coordinates of both SPs are positive. So, the cubic has exactly one root, at $x \approx 1/\varepsilon$. □

4873. The rate of change of angle, on the first positive edge, is least at $\theta = 0, 60°$, and greatest at the midpoint $\theta = 30°$. To find the rate of change of θ, we look at two elements:

- velocity perpendicular to OF,
- the distance $|OF|$.

Let the speed be 1. At the midpoint of an edge, the perpendicular component is of speed is 1. Just before a vertex, it is $\sqrt{3}/2$. The ratio of these is

$$1 : \tfrac{\sqrt{3}}{2}.$$

Let the distance from the centre to the midpoint be 1. At the midpoint, $|OF| = 1$. At the vertex, $|OF| = 2/\sqrt{3}$. The ratio of these is

$$1 : \tfrac{2}{\sqrt{3}}.$$

To combine these, consider the formula $l = r\theta$, in which l is arc length. Differentiated with respect to t, this is $v_{\text{perp}} = r\omega$, which is

$$\omega = \frac{v_{\text{perp}}}{r}.$$

So, the ratio of angular speeds is

$$1 : \tfrac{\sqrt{3}}{2} \div \tfrac{2}{\sqrt{3}}.$$

This simplifies to $4 : 3$, as required.

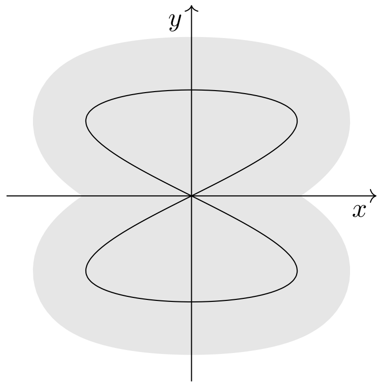
4874. Let $u = x^2$ and $v' = xe^{x^2}$, so that $u' = 2x$ and, integrating by inspection, $v = \frac{1}{2}e^{x^2}$. This gives

$$\begin{aligned} &\int x^3 e^{x^2}\,dx \\ &= \tfrac{1}{2}x^2e^{x^2} - \int xe^{x^2}\,dx \\ &= \tfrac{1}{2}x^2e^{x^2} - \tfrac{1}{2}e^{x^2} + c \\ &= \tfrac{1}{2}e^{x^2}(x^2 - 1) + c. \end{aligned}$$

4875. The centre moves with parametric equations

$$\begin{aligned} x &= 2\sin 2t, \\ y &= 2\sin t. \end{aligned}$$

These define a Lissajous curve. The set of (x, y) points through which the curve passes consists of all points within 1 unit distance of this curve:



4876. Consider prime factors:

- The product must contain three consecutive even numbers, of which one is a multiple of four. So, the product must be divisible by 2^4, but can't be guaranteed to be divisible by 2^5.
- The product must contain two consecutive multiples of three. So, it must be divisible by 3^2, but isn't necessarily divisible by 3^3.
- One multiple of five can be guaranteed. The product must be divisible by 5.
- One multiple of seven can be guaranteed. The product must be divisible by 7.
- No multiples of other, higher primes can be guaranteed to be present.

So, there is always a factor of

$$2^4 \times 3^2 \times 5 \times 7.$$

We can therefore guarantee divisibility by

$$2^a \times 3^b \times 5^c \times 7^d,$$

where the indices can be chosen, independently of one another, from

$$\begin{aligned} a &\in \{0, 1, 2, 3, 4\}, \\ b &\in \{0, 1, 2\}, \\ c &\in \{0, 1\}, \\ d &\in \{0, 1\}. \end{aligned}$$

Hence, the number of elements in K is

$$\begin{aligned} |K| &= 5 \times 3 \times 2 \times 2 \\ &= 60, \text{ as required.} \end{aligned}$$

4877. Let $y = \text{g}_a(x)$ be the equation of the tangent at $x = a$. The equation for intersections with the curve is

$$x^3 + 3x^2 - 5 - \text{g}_a(x) = 0.$$

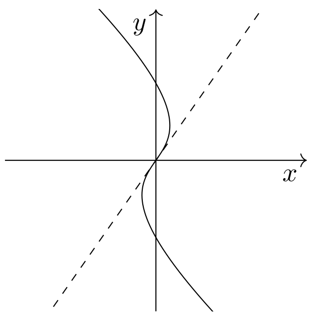
This is a cubic equation. We know, due to the point of tangency at $x = a$, that this equation has a repeated root at $x = a$, so a repeated factor of $(x - a)$. This factor could have multiplicity two, as in $(x-a)^2$ or three, as in $(x-a)^3$:

- If it has multiplicity two, then, taking out this quadratic factor leaves a linear factor, which corresponds to a distinct root. This is a point of re-intersection.
- If it has multiplicity three, then there can be no other points of intersection. This can only occur where, in the case of a cubic, the second derivative is zero*. This is at $x = 1$.

——— Nota Bene ———

To justify further the fact marked $*$, you can use contradiction. Assume that the second derivative is non-zero. Then, since the curvature is e.g. +ve in a small neighbourhood around $x = a$, the curve must remain e.g. above the tangent. This signifies a double, as opposed to triple root.

4878. In the vicinity of O, x and y are both small. In the limit, the RHS, being a cube, is negligible. So, the tangent at the origin is $ax + by = 0$. Solving for intersections of this tangent with the curve, $0 = (bx - ay)^3$. Since this has a triple root at O, the tangent must cross the curve. This signifies that the curve is inflected at the origin.



——— Nota Bene ———

The curve is a rotation of $y = x^3$.

4879. For fixed points,

$$\sum_{r=1}^{2k+1} x^r = x.$$

Writing this longhand,

$$\begin{aligned} & x + x^2 + x^3 + ... + x^{2k+1} = x \\ \Longrightarrow\ & x^2 + x^3 + ... + x^{2k+1} = 0 \\ \Longrightarrow\ & x^2\big(1 + x + ... + x^{2k-1}\big) = 0. \end{aligned}$$

So, $x = 0$ is a fixed point. The other option is

$$1 + x + ... + x^{2k-1} = 0.$$

This has a root at $x = -1$. Taking out the relevant factor,

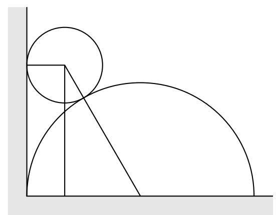
$$(1 + x)\big(1 + x^2 + x^4 + ... + x^{2k-2}\big) = 0.$$

This leaves

$$1 + x^2 + x^4 + ... + x^{2k-2} = 0.$$

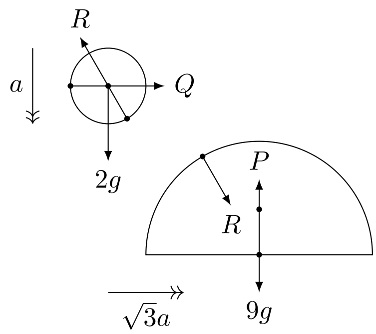
Every term in x has even degree, so the LHS is greater than or equal to 1. Hence, this equation has no real roots. The iteration I, therefore, has two fixed points, $x = 0$ and $x = -1$. □

4880. (a) Drawing in the radii etc., the scenario is



The line of centres has gradient $-\sqrt{3}$. Hence, the gradient of the common tangent is $1/\sqrt{3}$. So, at release, a small horizontal displacement δx in the half-cylinder will allow the cylinder to fall $\delta x/\sqrt{3}$. In the instantaneous limit, the ratio of displacements is $1 : \sqrt{3}$, so the ratio of accelerations is also $1 : \sqrt{3}$.

(b) The cross-sectional areas are in the ratio $2 : 9$, so let the masses be 2 kg and 9 kg. The force diagrams are



Resolving in the directions of the accelerations,

$$\begin{aligned} 2g - R\cos 30° &= 2a \\ R \sin 30° &= 9\sqrt{3}a. \end{aligned}$$

Solving these, $a = \frac{2}{29}g$ ms^{-2}.

4881. (a) A quarter-circle centred on O with radius r has equation $y = \sqrt{r^2 - x^2}$, where x takes values between 0 and r. Differentiating this,

$$\frac{dy}{dx} = -\frac{x}{\sqrt{r^2 - x^2}}.$$

So, using the arc length formula,

$$\begin{aligned} C &= 4\int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\ &= 4\int_0^r \sqrt{\frac{r^2}{r^2 - x^2}}\, dx \\ &= 4r\int_0^r \frac{1}{\sqrt{r^2 - x^2}}\, dx. \end{aligned}$$

(b) Let $x = r\sin\theta$, so that $dx = r\cos\theta\, d\theta$. The new limits are $\theta = 0$ to $\theta = \frac{\pi}{2}$. Enacting the substitution,

$$\begin{aligned} C &= 4r\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{r^2 - r^2\sin^2\theta}} \cdot r\cos\theta\, d\theta \\ &= 4r\int_0^{\frac{\pi}{2}} 1\, d\theta \\ &= 4r\Big[\theta\Big]_0^{\frac{\pi}{2}} \\ &\equiv 2\pi r, \text{ as required.} \end{aligned}$$

4882. The red counters have ended up in a group. So, the possibility space consists of the orders of

$$\{[\text{RRR}], \text{B}, \text{B}, \text{B}, \text{G}, \text{G}, \text{G}\}.$$

There are $\frac{7!}{3!3!} = 140$ orders of this. In successful outcomes, the blue counters also form a group:

$$\{[\text{RRR}], [\text{BBB}], \text{G}, \text{G}, \text{G}\}.$$

There are $\frac{5!}{3!} = 20$ ways of ordering this. So, the probability is $\frac{20}{140} = \frac{1}{7}$.

— Alternative Method —

Place the [RRR] group. We can place the first blue counter to the right, wlog: RRRB. Place the next two blue counters. Given that the [RRR] counters are a fixed block, the probability of ending up with RRRBBB is

$$\tfrac{2}{3} \times \tfrac{3}{4}.$$

We now place the greens, noting that the [RRR] group is inviolate, but the [BBB] group is not. There are five possible locations for the first G, of which three are successful. Continuing in this vein, the probability that the greens don't break up the blue group is

$$\tfrac{3}{5} \times \tfrac{4}{6} \times \tfrac{5}{7}.$$

So, the overall probability is

$$\tfrac{2}{3} \times \tfrac{3}{4} \times \tfrac{3}{5} \times \tfrac{4}{6} \times \tfrac{5}{7} = \tfrac{1}{7}.$$

4883. (a) i. The term $(x^2 + y)^2$ is non-negative, so $y^2 \le 1$. This gives $|y| \le 1$.

ii. Assume, for a contradiction, that $|x| > 2$. Then $x^2 > 4$, which means $x^2 + y > 3$, so $(x^2 + y)^2 > 9$. Since y^2 is non-negative, this is not possible.

(b) Differentiating implicitly,

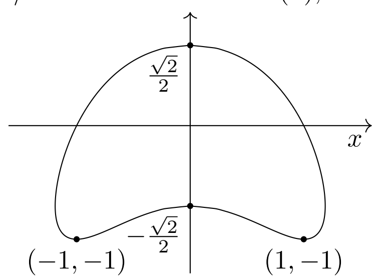
$$2(x^2 + y)\big(2x + \tfrac{dy}{dx}\big) + 2y\tfrac{dy}{dx} = 0.$$

Setting $\frac{dy}{dx} = 0$,

$$\begin{aligned} &2(x^2 + y)(2x) = 0 \\ \implies &x = 0 \text{ or } (x^2 + y) = 0. \end{aligned}$$

The former gives SPs at $(0, \pm\sqrt{2}/2)$. Subbing the latter into the curve equation, $y^2 = 1$, so there are SPs at $(1, \pm 1)$.

(c) The curve is a loop around the origin, as shown in part (a). It is symmetrical in the y axis. The x intercepts are ± 1 and the y intercepts are $\pm\sqrt{2}/2$. With the SPs from (b), this gives



4884. (a) Let the speed of the wheel be 1 unit/s. The coordinates of the centre C are then $(t, 1)$. The radial vector rotates clockwise at 1 rad/s, starting at $-\mathbf{j}$. In terms of time t, this is

$$\overrightarrow{CP} = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix}.$$

So, the position vector of P is

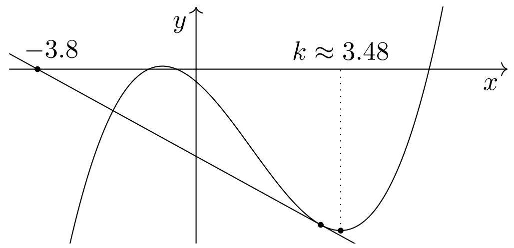
$$\begin{aligned} \overrightarrow{OP} &= \overrightarrow{OC} + \overrightarrow{CP} \\ &= \begin{pmatrix} t \\ 1 \end{pmatrix} + \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix} \\ &\equiv \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}. \end{aligned}$$

(b) Using the parametric integration formula,

$$\begin{aligned} A &= \int_0^{2\pi} y\frac{dx}{dt}\, dt \\ &= \int_0^{2\pi} (1 - \cos t)^2\, dt \\ &= \int_0^{2\pi} \tfrac{1}{2}\cos 2t - 2\cos t + \tfrac{3}{2}\, dt \\ &= \Big[\tfrac{1}{4}\sin 2t - 2\sin t + \tfrac{3}{2}t\Big]_0^{2\pi} \\ &= 3\pi. \end{aligned}$$

The wheel itself has area π. So, during one revolution, the area underneath the cycloid is three times the area of the wheel. QED.

4885. First, we sketch $y = \text{f}(x)$. The graph has two SPs at approximately $(-0.814, 1.46)$ and $(3.48, -77.8)$. So, the roots $x = a$ and $x = b$ are close together, one greater than and one less than $x = -0.814$. The curve is a positive cubic. The equation of the tangent at $x = 3$ is $y = -11x - 42$. This intercepts the x axis at $x = -\frac{42}{11} \approx -3.8$



There are three behaviours for $x_0 \geq 3$:

① For $x_0 \in [3, k)$, the tangent is shallower than that at $x = 3$. So, $x_1 < -3.8$. After this, the iteration must converge to $x = a$.

② For $x_0 = k$, $\text{f}'(x) = 0$: x_1 is undefined.

③ For $x_0 \in (k, \infty)$, the gradient is positive, and x_1 will be to the right of the root c. After this, the iteration must converge to $x = c$.

So, for no starting value $x_0 \geq 3$ will the iteration converge to the root $x = b$. □

4886. The left-hand bracket is

$$\begin{aligned} &\tan x + \cot x \\ \equiv\ &\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\ \equiv\ &\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \\ \equiv\ &\frac{1}{\sin x \cos x}. \end{aligned}$$

So, the LHS is

$$\begin{aligned} &\frac{\sin x + \cos x}{\sin x \cos x} \\ \equiv\ &\frac{1}{\cos x} + \frac{1}{\sin x} \\ \equiv\ &\sec x + \operatorname{cosec} x, \text{ as required.} \end{aligned}$$

4887. The sum of the first npq integers is

$$S_1 = \tfrac{1}{2}npq(npq + 1).$$

Of these, there are

① nq integers divisible by p, with sum

$$S_p = \tfrac{1}{2}npq(nq + 1).$$

② np integers divisible by q, with sum

$$S_q = \tfrac{1}{2}npq(np + 1).$$

③ n integers divisible by p and q, with sum

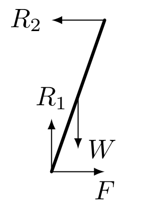
$$S_{pq} = \tfrac{1}{2}npq(n + 1).$$

S is given by $S_1 - S_p - S_q + S_{pq}$. Each term has a common factor of $\frac{1}{2}npq$. Taking this out,

$$\begin{aligned} S &= \tfrac{1}{2}npq\big(npq + 1 - (nq + 1) - (np + 1) + n + 1\big) \\ &\equiv \tfrac{1}{2}npq\big(npq - nq - np + n\big) \\ &\equiv \tfrac{1}{2}n^2pq\big(pq - q - p + 1\big) \\ &\equiv \tfrac{1}{2}n^2pq(p - 1)(q - 1), \text{ as required.} \end{aligned}$$

4888. The trig ratios are $\sin\theta = \frac{12}{13}$ and $\cos\theta = \frac{5}{13}$.

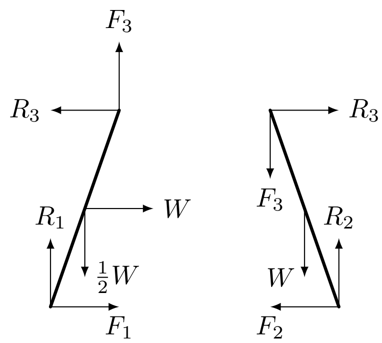
(a) The force diagram for the left-hand card is



Call the length 2. Taking moments around the base,

$$\begin{aligned} W \times \cos\theta &= R_2 \times 2\sin\theta \\ \Longrightarrow R_2 &= \tfrac{1}{2}W\cot\theta \\ &= \tfrac{5}{24}W. \end{aligned}$$

(b) With a force exerted on one card, there is no longer symmetry. Hence, friction acts between the cards. The force diagrams are



The surface is high friction, so F_1 and F_2 can be arbitrarily large. Slipping can only occur at the apex. Assuming limiting friction at that point, $F_3 = \mu R_3$. Taking moments around the base of the left-hand card,

$$\begin{aligned} &W(\cos\theta + \tfrac{1}{2}\sin\theta) = 2R_3\sin\theta + 2\mu R_3\cos\theta \\ \therefore\ \ &11W = (24 + 10\mu)R_3. \end{aligned}$$

Around the base of the right-hand card,

$$\begin{aligned} &W\cos\theta + 2\mu R_3\cos\theta = 2R_3\sin\theta \\ \therefore\ \ &5W = (24 - 10\mu)R_3. \end{aligned}$$

Solving simultaneously,

$$\begin{aligned} \frac{5W}{24 - 10\mu} &= \frac{11W}{24 + 10\mu} \\ \therefore\ \ 5(24 + 10\mu) &= 11(24 - 10\mu) \\ \Longrightarrow \mu &= \tfrac{9}{10}. \end{aligned}$$

So, for equilibrium, $\mu \geq \frac{9}{10}$, as required.

4889. Let $a_n = ar^{n-1}$ and $b_n = bs^{n-1}$. We are told that $a_n + b_n$ is also geometric. Equating ratios,

$$\begin{aligned}
&\frac{ar^2+bs^2}{ar+bs} = \frac{ar+bs}{a+b} \\
\implies &\left(ar^2+bs^2\right)(a+b) = (ar+bs)^2 \\
\implies &a^2r^2+abr^2+abs^2+b^2s^2 \\
&\qquad = a^2r^2+2abrs+b^2s^2 \\
\implies &abr^2-2abrs+abs^2 = 0 \\
\implies &ab(r-s)^2.
\end{aligned}$$

We are told that no term is zero. Hence, since a and b are non-zero, $r - s = 0$, and the common ratios of a_n and b_n are the same. □

4890. Let A start.

$$\begin{aligned}
&\mathbb{P}(A \text{ hits gold}) = p, \\
&\mathbb{P}(A \text{ fails to score}) = 1-p-q, \\
&\mathbb{P}(B \text{ hits gold}) = pq, \\
&\mathbb{P}(B \text{ fails to score}) = q(1-p-q).
\end{aligned}$$

If none of the above occur, then the competition starts again. So, consider the round in which the competition is decided:

$$\begin{aligned}
\mathbb{P}(A \text{ wins} \mid \text{a result}) &= \frac{p+q(1-p-q)}{1-q^2}, \\
\mathbb{P}(B \text{ wins} \mid \text{a result}) &= \frac{pq+(1-p-q)}{1-q^2}.
\end{aligned}$$

Equating these, we require

$$\begin{aligned}
&p+q(1-p-q) = pq+(1-p-q) \\
\implies &p+q-pq-q^2 = pq+1-p-q \\
\implies &q^2+2pq-2p-2q+1 = 0 \\
\implies &(q-1)(q+2p-1) = 0 \\
\implies &q = 1 \text{ or } q = 1-2p.
\end{aligned}$$

If $q = 1$, then the competition will never end. So, we require $q = 1 - 2p$.

4891. The equation of a generic normal is

$$y - a^2 = -\tfrac{1}{2a}(x-a).$$

The y intercept of this normal is $a^2 + 1/2$. Hence, in terms of a, the area of the region is given by the area of a trapezium minus the area under the curve:

$$\begin{aligned}
A &= \tfrac{1}{2}a\big(a^2+(a^2+1/2)\big) - \int_0^a x^2\,dx \\
&\equiv a^3 + \tfrac{1}{4}a - \tfrac{1}{3}a^3 \\
&\equiv \tfrac{2}{3}a^3 + \tfrac{1}{4}a.
\end{aligned}$$

Setting this to 1.452, we solve to find $a = 1.2$.

4892. Rename x as m.

The terms on the LHS are $\arctan m$, which is the angle of inclination of $y = mx$, and $\arctan 1/m$, which is the angle of inclination of $y = \frac{1}{m}x$. These lines have reciprocal gradients, so are reflections of one another in the line $y = x$, at inclination $\pi/4$. Hence, the sum of the two angles of inclination is $\frac{\pi}{2}$. This gives $\arctan x + \arctan 1/x \equiv \pi/2$. □

— Alternative Method —

Let $a = \arctan x$ and $b = \arctan 1/x$. Consider $\tan(a + b)$. By a compound-angle formula, this is

$$\begin{aligned}
\tan(a+b) &= \frac{\tan a + \tan b}{1-\tan a \tan b} \\
&= \frac{x+\frac{1}{x}}{1-x\cdot\frac{1}{x}}.
\end{aligned}$$

Since the denominator is 0, both sides must be undefined. We know that $a, b \in (0, \pi/2)$. The only sum of such numbers for which the tan function is undefined is $a + b = \pi/2$. Hence,

$$\arctan x + \arctan 1/x \equiv \pi/2, \text{ as required.}$$

4893. The first curve factorises as

$$y = x^2(x+1)(x-1).$$

This is a positive quartic with a double root at $x = 0$ and single roots at $x = \pm 1$. The second curve factorises as

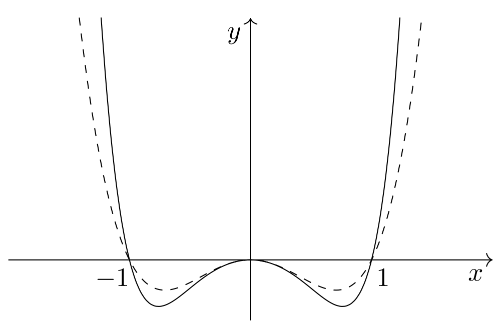
$$y = x^2(x^2+1)(x+1)(x-1).$$

This is a positive sextic also with a double root at $x = 0$ and single roots at $x = \pm 1$.

Looking for intersections between the curves,

$$\begin{aligned}
&x^4 - x^2 = x^6 - x^2 \\
\implies &x^6 - x^4 = 0 \\
\implies &x^4(x^2-1) = 0 \\
\implies &x^4(x+1)(x-1) = 0.
\end{aligned}$$

So, the curves are tangent (quadruple intersection) at the origin, and they cross (single intersections) at the intercepts $x = \pm 1$. The quartic is dashed in the diagram:



4894. Consider the suggested expressions. Firstly,

$$\begin{aligned}&(\sin 27^\circ + \cos 27^\circ)^2\\ =\ &\sin^2 27^\circ + 2\sin 27^\circ \cos 27^\circ + \cos^2 27^\circ\\ =\ &1 + \sin 54^\circ\\ =\ &\tfrac{1}{4}\big(5+\sqrt{5}\big).\end{aligned}$$

This gives

$$\sin 27^\circ + \cos 27^\circ = \tfrac{1}{2}\sqrt{5+\sqrt{5}}.$$

Secondly,

$$\begin{aligned}&(\sin 27^\circ - \cos 27^\circ)^2\\ =\ &\sin^2 27^\circ - 2\sin 27^\circ \cos 27^\circ + \cos^2 27^\circ\\ =\ &1 - \sin 54^\circ\\ =\ &\tfrac{1}{4}\big(3-\sqrt{5}\big).\end{aligned}$$

This gives

$$\sin 27^\circ - \cos 27^\circ = \tfrac{1}{2}\sqrt{3-\sqrt{5}}.$$

Doubling the equations and adding them,

$$4\sin 27^\circ = \sqrt{5+\sqrt{5}} + \sqrt{3-\sqrt{5}}, \text{ as required.}$$

4895. We can derive the relevant product-to-sum formula as follows:

$$\begin{aligned}&\cos(p+q) - \cos(p-q) = 2\sin p \sin q\\ \Longrightarrow\ &\sin p \sin q = \tfrac{1}{2}\big(\cos(p+q) + \cos(p-q)\big).\end{aligned}$$

So, the integral is

$$\begin{aligned}&\int_0^{2\pi} \sin ax \sin bx\, dx\\ =\ &\tfrac{1}{2}\int_0^{2\pi} \cos(a+b)x + \cos(a-b)x\, dx\\ =\ &\tfrac{1}{2}\Big[\sin\big((a+b)x\big) + \sin\big((a-b)x\big)\Big]_0^{2\pi}\\ =\ &\tfrac{1}{2}\Big(\sin\big(2\pi(a+b)\big) + \sin\big(2\pi(a-b)\big)\Big).\end{aligned}$$

Since a, b are natural numbers, $a+b$ and $a-b$ are integers. So, both terms in the above are zero, because, for $n \in \mathbb{Z}$, $\sin(2n\pi) = 0$. This gives the required result:

$$\int_0^{2\pi} \sin ax \sin bx\, dx = 0.$$

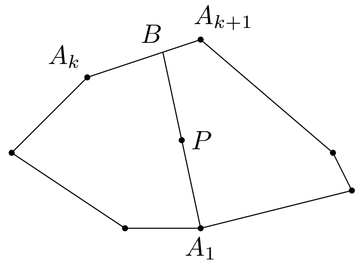
— Nota Bene —

The above result can also be shown directly by symmetry, using the fact that $\sin ax$ and $\sin bx$ are periodic, with periods dividing 2π. In some sense, this makes the result "obvious". It is quite difficult, however, to make such an argument rigorous and thus convincing to those who don't yet understand it. The algebraic approach is better.

4896. We prove the result by construction. Draw a line through point P and vertex A_1. If this line passes through another vertex A_k, then we have the result immediately, as a point on A_1A_k can be written

$$\mathbf{p} = \lambda \mathbf{a}_1 + (1-\lambda)\mathbf{a}_k.$$

Otherwise, the line passes through an edge. Call this A_kA_{k+1} and the point of intersection B.



The position vector of B may be expressed as

$$\mathbf{b} = \lambda \mathbf{a}_k + (1-\lambda)\mathbf{a}_{k+1}.$$

So, the position vector of P is

$$\begin{aligned}\mathbf{p} &= \mu \mathbf{a}_1 + (1-\mu)\mathbf{b}\\ &= \mu \mathbf{a}_1 + (1-\mu)\big(\lambda \mathbf{a}_k + (1-\lambda)\mathbf{a}_{k+1}\big)\\ &= \mu \mathbf{a}_1 + \lambda(1-\mu)\mathbf{a}_k + (1-\lambda)(1-\mu)\mathbf{a}_{k+1}.\end{aligned}$$

So, to set up the final result, set all constants λ_i to zero, other than

$$\begin{aligned}\lambda_1 &= \mu,\\ \lambda_k &= \lambda(1-\mu),\\ \lambda_{k+1} &= (1-\lambda)(1-\mu).\end{aligned}$$

This gives $\mathbf{p} = \displaystyle\sum_{i=1}^{n} \lambda_i \mathbf{a}_i$. Furthermore,

$$\begin{aligned}\sum \lambda_i &= \lambda_1 + \lambda_k + \lambda_{k+1}\\ &= \mu + \lambda(1-\mu) + (1-\lambda)(1-\mu)\\ &\equiv \mu + 1 - \mu\\ &\equiv 1, \text{ as required.}\end{aligned}$$

4897. The LHS of the proposed equation is

$$\begin{aligned}&4T_n^3 - T_n^2 + 3(n+1)^5\\ =\ &\tfrac{1}{2}n^3(n+1)^3 - \tfrac{1}{4}n^2(n+1)^2 + 3(n+1)^5\\ \equiv\ &\tfrac{1}{4}(n+1)^2\big(2n^3(n+1) - n^2 + 12(n+1)^3\big).\end{aligned}$$

The quartic factor is

$$\begin{aligned}&2n^4 + 2n^3 - n^2 + 12n^3 + 36n^2 + 36n + 12\\ \equiv\ &2n^4 + 14n^3 + 35n^2 + 36n + 12\\ \equiv\ &(n+2)^2(2n^2 + 6n + 3).\end{aligned}$$

So, the LHS is

$$\tfrac{1}{4}(n+1)^2(n+2)^2(2n^2 + 6n + 3).$$

Starting again with the RHS,

$$\begin{aligned}&4T_{n+1}^3 - T_{n+1}^2\\ &= \tfrac{1}{2}(n+1)^3(n+2)^3 - \tfrac{1}{4}(n+1)^2(n+2)^2\\ &\equiv \tfrac{1}{4}(n+1)^2(n+2)^2\big(2(n+1)(n+2)-1\big)\\ &\equiv \tfrac{1}{4}(n+1)^2(n+2)^2\big(2n^2+6n+3\big).\end{aligned}$$

So, the algebra of the proof holds.

— Nota Bene —

The above is an *inductive step*. When combined with a *base case* and some logical structure, it can be turned into a formal *proof by induction*.

4898. We classify the restricted possibility space by the order of rotational symmetry.

④ Order 4. There are four orientations for the tile in the top left. Rotating the whole shape through $90°, 180°, 270°$ fixes the orientations of the other three. This gives four outcomes, two of which are shown below:

② Order 2. There are four options for the tile in the top left, as before. This fixes the tile in the bottom right. The orientations of the other two tiles are bound to each other, which would give 4 possibilities. However, one of these produces rotational symmetry of order 4. So, there are $4 \times 3 = 12$ outcomes which have rotational symmetry order 2.

The possibility space consists of the 16 outcomes with rotational symmetry. Hence, the probability that the symmetry is order 4 is $1/4$.

4899. Let $u = \cos x$, so that $du = -\sin x dx$.

$$\begin{aligned}&\int \frac{1}{\sin^3 x - \sin x}\,dx\\ =&\int \frac{1}{\sin^2 x(1-\sin^2 x)} \cdot -\sin x\,dx\\ =&\int \frac{1}{(1-u^2)u^2}\,du.\end{aligned}$$

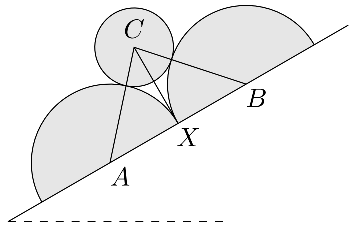
Writing the integrand in partial fractions,

$$\begin{aligned}&\frac{1}{(1-u)(1+u)u^2}\\ \equiv\ &\frac{1}{u^2} + \frac{1}{2(u+1)} - \frac{1}{2(u-1)}.\end{aligned}$$

So, the integral is

$$\begin{aligned}&\int \frac{1}{u^2} + \frac{1}{2(u+1)} - \frac{1}{2(u-1)}\,du\\ =&-u^{-1} + \tfrac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + c\\ =&\ \tfrac{1}{2}\ln\left|\frac{\cos x+1}{\cos x-1}\right| - \sec x + c.\end{aligned}$$

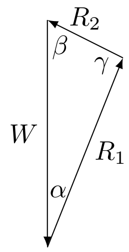
4900. (a) The scenario is



Triangle ABC has side lengths $(3r, 3r, 4r)$, so

$$\angle ACX = \angle BCX = \arcsin {}^2\!/\!_3.$$

Hence, the angle between AC and the vertical is $\alpha = \arcsin {}^2\!/\!_3 - 30°$, and the angle between BC and the vertical is $\beta = \arcsin {}^2\!/\!_3 + 30°$. The last angle is duly $\gamma = 180° - 2\arcsin {}^2\!/\!_3$. The triangle of forces is as follows:



(b) Using the sine rule,

$$\begin{aligned}R_1 &= \frac{\sin\beta}{\sin\gamma}W\\ &= \frac{\sin(\arcsin {}^2\!/\!_3 + 30°)}{\sin(180° - 2\arcsin {}^2\!/\!_3)}W.\end{aligned}$$

Using $\cos(\arcsin {}^2\!/\!_3) = \sqrt{5}/3$, the denominator simplifies as

$$\begin{aligned}&\sin(180° - 2\arcsin {}^2\!/\!_3)\\ =&\ \sin(2\arcsin {}^2\!/\!_3)\\ =&\ 2\sin(\arcsin {}^2\!/\!_3)\cos(\arcsin {}^2\!/\!_3)\\ =&\ \tfrac{4\sqrt{5}}{9}.\end{aligned}$$

The numerator simplifies as

$$\begin{aligned}&\sin(\arcsin {}^2\!/\!_3 + 30°)\\ =&\ \sin(\arcsin {}^2\!/\!_3)\cos 30° + \cos(\arcsin {}^2\!/\!_3)\sin 30°\\ =&\ \tfrac{\sqrt{3}}{3} + \tfrac{\sqrt{5}}{6}.\end{aligned}$$

So, the final result is

$$\begin{aligned}R_1 &= \frac{\frac{\sqrt{3}}{3} + \frac{\sqrt{5}}{6}}{\frac{4\sqrt{5}}{9}}W\\ &\equiv \left(\tfrac{3}{8} + \tfrac{3}{4}\sqrt{\tfrac{3}{5}}\right)W, \text{ as required.}\end{aligned}$$

— End of 49th Hundred —